

Localization for Khovanov homologies:

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LOCALIZATION FOR KHOVANOV HOMOLOGIES

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LOCALIZATION FOR KHOVANOV HOMOLOGIES

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In 2010, Seidel and Smith used their localization framework for Floer homologies to prove a Smith-type rank inequality for the symplectic Khovanov homology of 2-periodic links in the 3-sphere. Hendricks later used similar geometric techniques to prove analogous rank inequalities for the knot Floer homology of 2-periodic links. We use combinatorial and space-level techniques to prove analogous Smith-type inequalities for various flavors of Khovanov homology for periodic links in the 3-sphere of any prime periodicity.

First, we prove a graded rank inequality for the annular Khovanov homology of 2-periodic links by showing grading obstructions to longer differentials in a localization spectral sequence. We remark that the same method can be extended to p -periodic links.

Second, in joint work with Matthew Stoffregen, we construct a \mathbb{Z}/p -equivariant stable homotopy type for odd and even, annular and non-annular Khovanov homologies, using Lawson, Lipshitz, and Sarkar's Burnside functor construction of a Khovanov stable homotopy type. Then, we identify the fixed-point sets and apply a version of the classical Smith inequality to obtain spectral sequences and rank inequalities relating the Khovanov homology of a periodic link with the annular Khovanov homology of the quotient link. As a corollary, we recover a rank inequality for Khovanov homology conjectured by Seidel and Smith's work on localization and symplectic Khovanov homology.

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Chapter 1

Introduction

This dissertation is essentially the concatenation of two articles resulting from studying the following question, which arises from Seidel and Smith’s work on localization in Floer homologies [SS10] and on symplectic Khovanov homology [SS06]: *for a link \tilde{L} in S^3 with $\mathbb{Z}/2$ rotational symmetry, is the Khovanov homology of \tilde{L} at least as large as that of its quotient link L ?*

The context of this problem is discussed in the introductions of the two attached articles, which I encourage the reader to look over, as they not only provide an idea of the landscape of localization problems in categorified invariants in low-dimensional topology but also paint a picture of how this simple question about periodic links and the rank of Khovanov homology lead me to engage with various tools.

In response to the question, in this dissertation, we show the following.

1. *Yes, the rank inequality holds.*

Seidel and Smith applied their localization framework for Floer homologies [SS10] to their geometrically defined symplectic Khovanov homology Kh_{symp} [SS06], which they conjectured is isomorphic to (combinatorial) Khovanov homology Kh [Kho00], to obtain the rank inequality for Kh_{symp} (over \mathbb{F}_2). Since Kh is inherently combi-

natorial (the definition of the invariant is a recipe for how to compute it, granted infinite patience), one would expect a proof of the Kh rank inequality to not require using an isomorphism to a geometrically-defined invariant.

2. *It holds because of spectral sequence involving annular Khovanov homology.*

This was surprising, but in hindsight should be expected. By the Smith Conjecture, the fixed-point set of any orientation-preserving diffeomorphism of S^3 must be empty, two points, or an unknot. The algebraic structures representing the topological phenomenon reflects the presence of this axis.

3. *In fact, it also holds for \mathbb{Z}/p actions, for any prime p .*

The analogue to the $\mathbb{Z}/2$ -periodic result in knot Floer homology is due to Hendricks [Hen15]. Due to inherent restrictions coming from Heegaard Floer diagrams, the knot Floer argument can only be used for $\mathbb{Z}/2$ actions. (Nevertheless, she reproves many classical results for p^r -periodic knots, by analyzing Heegaard diagrams for these.) In the Khovanov case, the proofs for $\mathbb{Z}/2$ are not that different from the more general \mathbb{Z}/p proofs.

4. *A relatively low-tech proof suffices to show the analogous rank inequality for annular Khovanov homology.*

The extra annular grading in annular Khovanov homology is enough to obstruct longer differentials in the spectral sequence. Though Chapter 2 only handles the 2-periodic case, observe that the argument that longer differentials vanish extends to p -periodic links, via the analysis explained in Chapter 3, in the proof of Proposition 3.76.

This dissertation is organized as follows. Chapter 2 consists of the article “A rank inequality for the annular Khovanov homology of 2-periodic links,” published in *Algebraic*

and *Geometric Topology*, Volume 18, Issue 2 [Zha18]. A conjecture from Section 2.8 led to the work with Matthew Stoffregen in Chapter 3, which consists of the article “Localization in Khovanov homology.” At present, this article has been submitted for publication and is available as an arXiv preprint [SZ18]. No attempt has been made to reconcile the differences in notation used in the two papers. The two papers are each rather self-contained, so notational differences should be self-explanatory. Nevertheless, it is worth mentioning a few key differences.

- The cube $\{0, 1\}^n$ plays an important role in both papers. In Chapter 2, I define a cubes of resolutions (Definition 2.13) and of chains (Definition 2.14), and notationally distinguished them by font. In Chapter 3, Matt and I use the more elegant notation of describing all such objects as functors from the cube category (Section 3.2.1). But because Khovanov homology is actually defined as a cohomological theory, the cube category in Chapter 3 is actually the opposite of the cube category implicit in Chapter 2. In other words, Chapter 2 works entirely with the cohomological complex, whereas Chapter 3 studies a homotopy type.
- In Chapter 2, the homological, quantum, and winding number gradings are associated to the letters i, j , and k , respectively. In Chapter 3, we use the letters h, q , and k , respectively.
- The Khovanov chain complex is denoted CKh in Chapter 2, and Kc in Chapter 3. The difference in notation is analogous for the annular theory. (Is “Kh” short for “Khovanov” or is it an acronym for “Khovanov homology?”)

I encourage undergraduate and graduate students who are interested in hands-on understanding and manipulation of spectral sequences coming from filtrations to take a look at the algebraic preliminaries in Chapter 2.

Chapter 2

A rank inequality for the annular Khovanov homology of 2-periodic links

2.1 Introduction

The world of topological objects is rich with spaces exhibiting interesting symmetries. These symmetries manifest themselves in the algebraic invariants we use to study the objects and can be manipulated to provide insights into the object itself. As fundamental, ubiquitous objects in low-dimensional topology, knots and links are some of the most natural objects to have algebraic invariants associated to them.

The rise of homology-type invariants categorifying polynomial link invariants have led to the study of equivariant homology theories by Chbili [[Chb10](#)], Seidel and Smith [[SS10](#)], Hendricks [[Hen12](#)] [[Hen15](#)], Politarczyk [[Pol15](#)] [[Pol17](#)], Borodzik and Politarczyk [[BP17](#)], and many others, building and improving upon earlier work of Murasugi [[Mur71](#)] [[Mur88](#)], Yokota [[Yok93](#)], and Przytycki [[Prz98](#)] relating the polynomial invariants of

periodic links with those of their quotient links.

A link \tilde{L} in S^3 is p -periodic if there is a p -periodic automorphism τ of S^3 with fixed set an unknot B restricting to a diffeomorphism on \tilde{L} ; the *quotient link* L is the quotient of \tilde{L} under the action of τ . Since \tilde{L} can naturally be viewed in a thickened annulus $A \times I$ by deleting the fixed set B from S^3 , Asaeda, Przytycki, and Sikora's *annular Khovanov homology* [APS04] seems particularly suitable for probing periodic links. We refer to a link in the thickened annulus as an *annular link*.

In this article, we construct a Tate-like bicomplex (§2.4) to show rank inequalities for the annular Khovanov homology of a 2-periodic annular link \tilde{L} and that of its quotient link L . For any annular link L , let $AKh^{j,k}(L)$ denote the annular Khovanov homology of L at quantum grading level j and \mathfrak{sl}_2 weight space grading level k .

Theorem 2.1. *Let \tilde{L} be a 2-periodic link with quotient link L . For each pair of integers (j, k) , there is a spectral sequence with*

$$E^1 \cong AKh^{2j-k,k}(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}] \Rightarrow E^\infty \cong AKh^{j,k}(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

A rank inequality immediately follows:

Corollary 2.2. *For \tilde{L} and L as above,*

$$rk_{\mathbb{F}_2} AKh^{j,k}(L) \leq rk_{\mathbb{F}_2} AKh^{2j-k,k}(\tilde{L}).$$

This result generalizes [Cor16], where Cornish proves a rank inequality between the next-to-top \mathfrak{sl}_2 weight space gradings in the annular Khovanov homologies of a 2-periodic link and that of its quotient by employing the Lipshitz-Treumann spectral sequence [LT16] with Auroux, Grigsby, and Wehrli's identification of the next-to-top winding grading with the Hochschild homology of a Khovanov-Seidel bimodule [AGW15]. He remarks that by using Beliakova-Putyra-Wehrli's generalization of Auroux, Grigsby, and

Wehrli's work [BPW19], one should be able to prove similar rank inequalities in other \mathfrak{sl}_2 weight space gradings, but that this requires checking the Lipshitz-Treumann algebraic conditions for larger dg algebras.

While our result does not require the links to be braid closures, annular Khovanov homology is often used to study braids, as the set of conjugacy classes of a braid group B_n embed into the set of isotopy classes of annular links. In particular, B_n can be viewed as the mapping class group of a disk with n punctures, and Corollary 2.2 suggests that the rank of annular Khovanov homology could provide a measure of complexity in the mapping class group.

Following the notation in [Rob13] (page 418, under Definition 2.2; also see §2.7 in this article), the decategorification of Theorem 2.1 can be written as follows:

Corollary 2.3. *For all j and k ,*

$$\langle q_{k,\tilde{L}}(-1, q), q^{2j-k} \rangle \equiv \langle q_{k,L}(-1, q), q^j \rangle \pmod{2},$$

where $\langle f, g \rangle$ denotes the coefficient of g in f .

However, the spectral sequence hints that the grading $j_1 := j - k$ is actually more pertinent to annular links (see [GLW18] and [GLW17] for more evidence). With this in mind, one can write down the decategorification as a statement similar to Murasugi's theorem on the Jones polynomial of periodic links [Mur88]:

Corollary 2.4. $V_{\tilde{L}}(1, q, q^{-1}) \equiv [V_L(1, q, q^{-1})]^2 \pmod{2}.$

The Jones polynomial of a periodic link \tilde{L} is related to the decategorification of the annular Khovanov homology of its quotient link L in the following way:

Corollary 2.5. $V_{\tilde{L}}(1, q, 1) \equiv V_L(1, q^2, q^{-1}) \pmod{2}.$

Moreover, we conjecture that a similar spectral sequence also relates the Khovanov homology [Kho00] of \tilde{L} with the *annular* Khovanov homology of L (see §2.8):

Conjecture 2.6. *Let \tilde{L} be a 2-periodic link in S^3 with quotient link L . There is a spectral sequence with*

$$E^1 \cong Kh(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}] \Rightarrow E^\infty \cong AKh(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

This would in turn imply the following cascade of rank inequalities:

$$rk_{\mathbb{F}_2} AKh(\tilde{L}) \geq rk_{\mathbb{F}_2} Kh(\tilde{L}) \geq rk_{\mathbb{F}_2} AKh(L) \geq rk_{\mathbb{F}_2} Kh(L)$$

where the first and third inequalities are due to Roberts' spectral sequence coming from the \mathfrak{sl}_2 weight space filtration on $CKh(\mathcal{D}(\tilde{L}))$ and $CKh(\mathcal{D}(L))$ [Rob13].

Note that this conjecture predicts much more than a rank inequality for Khovanov homology. In many contexts, spectral sequences from the homology of an associated graded object like AKh to the homology of the filtered complex tends to dramatically decrease rank, so it is surprising to find so much evidence for the middle inequality $rk_{\mathbb{F}_2} Kh(\tilde{L}) \geq rk_{\mathbb{F}_2} AKh(L)$. In fact, Corollary 2.5 would be the decategorification of Conjecture 2.6. We can show that the conjecture and consequent rank inequalities hold for the following family of annular links:

Theorem 2.7. *If the annular braid closure $L = \widehat{\beta}$ has a diagram with at most 1 positive crossing, then the spectral sequence in Conjecture 2.6 exists and the cascade of rank inequalities holds.*

Furthermore, it follows from Theorem 2.7 that the cascade of rank inequalities also holds for positive braid closures (and more):

Corollary 2.8. *If the annular braid closure $L = \widehat{\beta}$ has a diagram with at most 1 negative crossing, then the cascade of rank inequalities holds.*

We conclude by discussing frameworks for obstructing 2-periodicity in links which arise from our results.

2.1.1 Acknowledgements

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2.2 Algebraic preliminaries

We begin by discussing the necessary homological algebra. The main theorem follows from computing two related spectral sequences arising from a particular bicomplex (constructed in §2.4). Spectral sequences from bicomplexes are a special case of spectral sequences from filtered complexes, which we discuss first.

The spectral sequences will be computed explicitly by making use of a well-known *cancellation lemma* (Lemma 2.10). Due to a special property of our complexes (see Definition 2.9), the algebraic computations can be described visually using dots and arrows; we describe this at the end of §2.2.1.

In §2.2.5, we give a brief overview of the ideas in Borel equivariant cohomology which motivate the constructions in this article.

2.2.1 Filtered chain complexes

Let \mathcal{C}_\bullet be a chain complex of \mathbb{F}_2 vector spaces, with differential ∂ .

A *decreasing \mathbb{Z} -filtration* of \mathcal{C}_\bullet is a sequence of subcomplexes indexed by decreasing integers:

$$\dots \subset F_n \mathcal{C}_\bullet \subset F_{n-1} \mathcal{C}_\bullet \subset F_{n-2} \mathcal{C}_\bullet \subset \dots \subset F_m \mathcal{C}_\bullet \subset \dots \mathcal{C}_\bullet.$$

The filtration $F_* \mathcal{C}_\bullet$ provides a *filtration grading* for the chains in \mathcal{C}_\bullet : $x \in \mathcal{C}_\bullet$ has filtration grading p if and only if $x \in F_p \mathcal{C}_\bullet$ and $x \notin F_{p+1} \mathcal{C}_\bullet$. The *associated graded vector space* $G_* \mathcal{C}_\bullet$ is given by

$$G_p \mathcal{C}_\bullet = F_p \mathcal{C}_\bullet / F_{p+1} \mathcal{C}_\bullet.$$

The existence of the filtration on \mathcal{C}_\bullet implies that every component of ∂ originating from $x \in \mathcal{C}_\bullet$ maps to targets y where the filtration grading of y is at least that of x ; we say that ∂ is non-negative with respect to the filtration grading.

Let \mathcal{H}_\bullet denote the homology of the chain complex \mathcal{C}_\bullet . The filtration $F_* \mathcal{C}_\bullet$ induces a filtration $F_* \mathcal{H}_\bullet$ on \mathcal{H}_\bullet : the class $[x] \in \mathcal{H}_\bullet$ is in $F_p \mathcal{H}_\bullet$ if and only if it is represented by some chain $x \in [x]$ where $x \in F_p \mathcal{C}_\bullet$. The induced filtration gives rise to the associated graded vector space for homology, $G_* \mathcal{H}_\bullet$.

Definition 2.9. In our work, the underlying vector space of \mathcal{C}_\bullet is always freely generated by a finite collection of distinguished generators which come equipped with some \mathbb{Z} -grading gr , and the differential is non-negative with respect to gr . In this case, gr induces a \mathbb{Z} -filtration and takes on the role of the filtration grading. We call the distinguished generators a *filtered graded basis*. A component ∂_r of the differential *shifts the filtration grading by $s \in \mathbb{Z}$* or is of *degree r with respect to the filtration grading* if for all filtered graded basis elements x , $\partial_s(x)$ is a linear combination of basis elements in grading $gr(x) + s$.

Similarly, a bigraded underlying vector space induces a $\mathbb{Z} \oplus \mathbb{Z}$ -filtration if the differ-

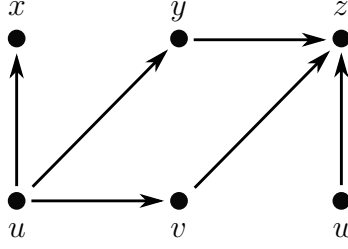


Figure 2.1

ential is monotone with respect to each grading.

Let (\mathcal{C}, ∂) be such a complex. We can visualize (\mathcal{C}, ∂) by using dots to represent the distinguished generators and arrows to represent components of the differential, as shown in Figure 2.1. We write $\partial(x, y)$ for the coefficient of y in $\partial(x)$, so that $\partial(x)$ is the sum of $\partial(x, y) \cdot y$ as y runs over all the distinguished generators of \mathcal{C} . If $\partial(x, y) = 1$, we draw an arrow from x to y .

2.2.2 Spectral sequences from filtered complexes

Spectral sequences are used to gradually approximate the true homology of a complex by gradually simplifying the complex, whilst preserving the homology. We execute this by repeatedly applying the *cancellation* operation (Figure 2.2), which relies on the following well-known “cancellation lemma.”

Lemma 2.10. *[[Bal11], Lemma 4.1]*

Let (\mathcal{C}, ∂) be a complex of \mathbb{F}_2 vector spaces freely generated by elements x_i . Let $\partial(x_i, x_j)$ be the coefficient of x_j in $\partial(x_i)$. If $\partial(x_k, x_l) = 1$, define a new complex $(\mathcal{C}', \partial')$ with generators $\{x_i \mid i \neq k, l\}$ and differential

$$\partial'(x_i) = \partial(x_i) + \partial(x_i, x_l)\partial(x_k).$$

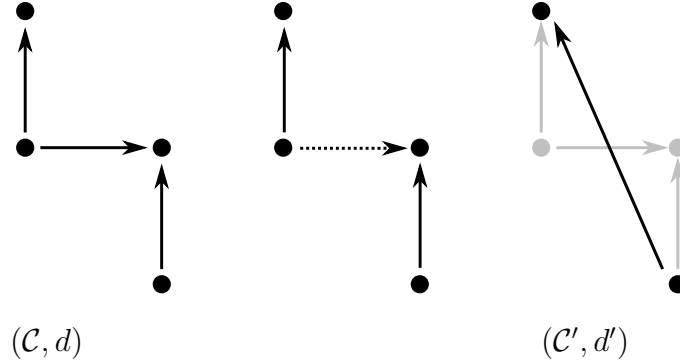


Figure 2.2: The cancellation process. On the left is the original complex. In the middle, the dotted arrow represents the arrow about to be cancelled. On the right is the resulting complex, with fewer generators and arrows, but with the same homology as the original complex.

Then $(\mathcal{C}', \partial')$ is chain homotopy equivalent to (\mathcal{C}, ∂) .

Given a decreasing \mathbb{Z} -filtered complex $F_*\mathcal{C}_\bullet$ with filtered differential ∂ , we can build a spectral sequence to compute the homology by starting with the associated graded complex and sequentially cancelling components of the differential of larger and larger filtration degree.

The pages of our spectral sequence are denoted (E^r, d^r) , where E_p^r is the vector space at filtration grading p surviving to page r , $E^r = \bigoplus_p E_p^r$, and d^r is the induced differential on E^r induced by cancellation on the previous pages. For example, $E_p^0 = G_p\mathcal{C}/G_{p-1}\mathcal{C}$, and d^0 is the sum of all the components of ∂ which preserve the filtration grading gr . The next page (E^1, d^1) is obtained by canceling all the components of d^0 , and is chain homotopy equivalent to (E^0, d^0) ; the underlying vector space \mathcal{C}' injects into \mathcal{C} and therefore has an induced filtration.

Iterating this process, we obtain pages (E^r, d^r) for all $r \geq 0$. In some situations, the spectral sequence eventually *collapses*, i.e. there is some N such that for all $r \geq N$, $d^r = 0$, and all pages are identical from then on. In such cases, the limit term $E^\infty := E^N = E^{N+1} = \dots$ is identified with the homology of the original complex (\mathcal{C}, ∂) . Figure

2.3 shows the cancellation process (the spectral sequence) used to compute the homology of a filtered complex.

2.2.3 Bicomplexes

A *bicomplex* $\mathcal{C}_{\bullet,\bullet}$ (Figure 2.4) is a bigraded \mathbb{F}_2 vector space with differentials

$$\partial_{p,q}^h : \mathcal{C}_{p,q} \rightarrow \mathcal{C}_{p+1,q}$$

$$\partial_{p,q}^v : \mathcal{C}_{p,q} \rightarrow \mathcal{C}_{p,q+1}$$

such that $\partial^h \circ \partial^v + \partial^v \circ \partial^h = 0$. Here ∂^h is the sum of all of the *horizontal differentials* $\partial_{p,q}^h$ and ∂^v is the sum of all of the *vertical differentials* $\partial_{p,q}^v$.

The corresponding *total complex* $Tot(\mathcal{C})_{\bullet}$ is given by

$$Tot(\mathcal{C})_n = \bigoplus_{p+q=n} \mathcal{C}_{p,q}$$

with differential $\partial^{Tot} = \partial^h + \partial^v$.

The total complex has two standard filtrations vF (the *row-wise filtration*) and hF (the *column-wise filtration*) induced by the rows and columns of the bicomplex, respectively, defined below, and depicted in Figure 2.5:

$$({}^vF_m \mathcal{C})_{p,q} = \begin{cases} \mathcal{C}_{p,q} & \text{if } q \leq m \\ 0 & \text{otherwise} \end{cases}$$

$$({}^hF_m \mathcal{C})_{p,q} = \begin{cases} \mathcal{C}_{p,q} & \text{if } p \leq m \\ 0 & \text{otherwise} \end{cases}.$$

With respect to either of these filtrations, the total differential ∂^{Tot} is non-negative.

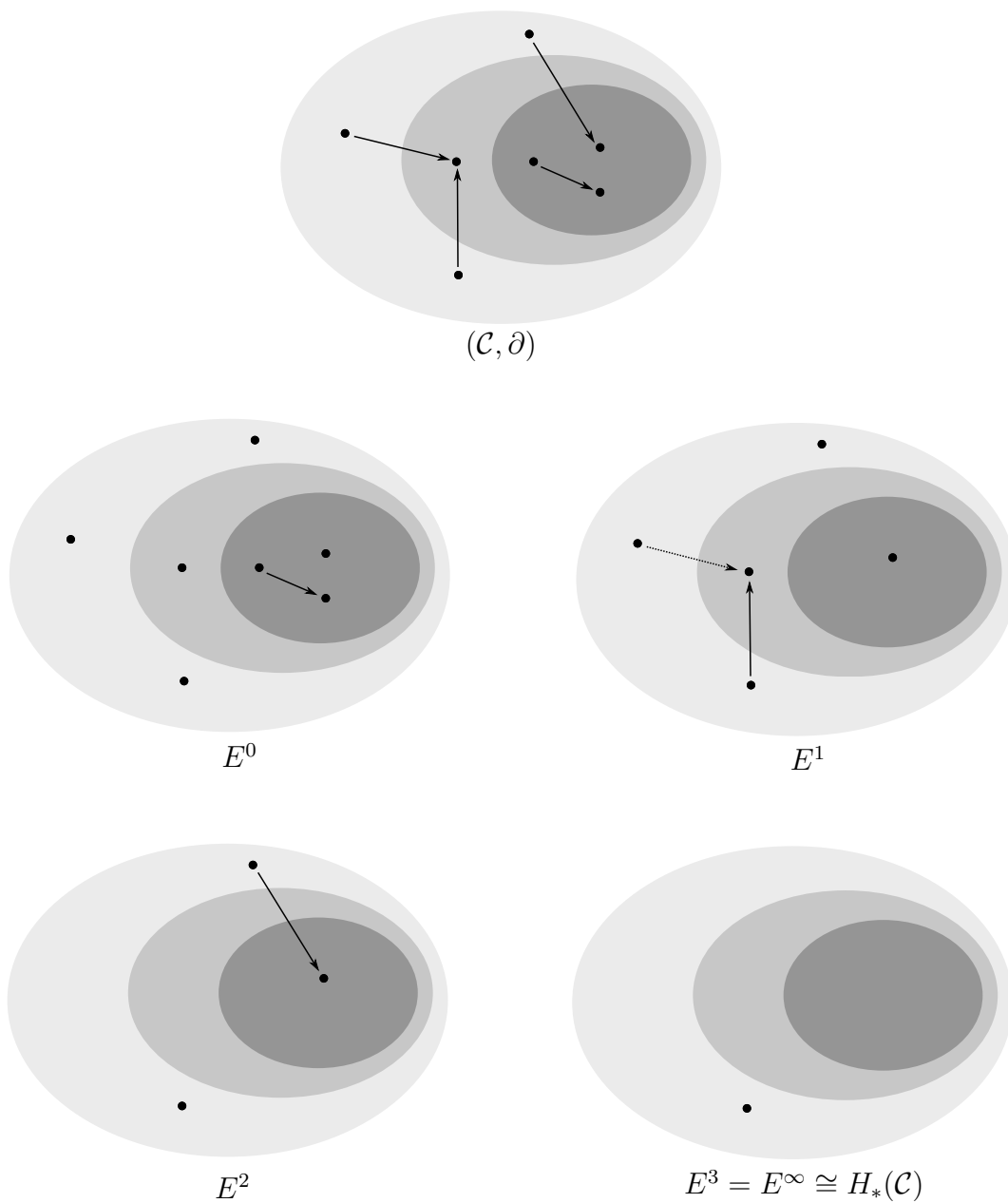


Figure 2.3: The spectral sequence which computes the homology of a filtered complex.

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \vdots \\
& & \uparrow \partial^v & & \uparrow \partial^v & & \uparrow \partial^v \\
\cdots & \xrightarrow{\partial^h} & \mathcal{C}_{p-1,q+1} & \xrightarrow{\partial^h} & \mathcal{C}_{p,q+1} & \xrightarrow{\partial^h} & \mathcal{C}_{p+1,q+1} \xrightarrow{\partial^h} \cdots \\
& & \uparrow \partial^v & & \uparrow \partial^v & & \uparrow \partial^v \\
\cdots & \xrightarrow{\partial^h} & \mathcal{C}_{p-1,q} & \xrightarrow{\partial^h} & \mathcal{C}_{p,q} & \xrightarrow{\partial^h} & \mathcal{C}_{p+1,q} \xrightarrow{\partial^h} \cdots \\
& & \uparrow \partial^v & & \uparrow \partial^v & & \uparrow \partial^v \\
\cdots & \xrightarrow{\partial^h} & \mathcal{C}_{p-1,q-1} & \xrightarrow{\partial^h} & \mathcal{C}_{p,q-1} & \xrightarrow{\partial^h} & \mathcal{C}_{p+1,q-1} \xrightarrow{\partial^h} \cdots \\
& & \uparrow \partial^v & & \uparrow \partial^v & & \uparrow \partial^v \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

Figure 2.4

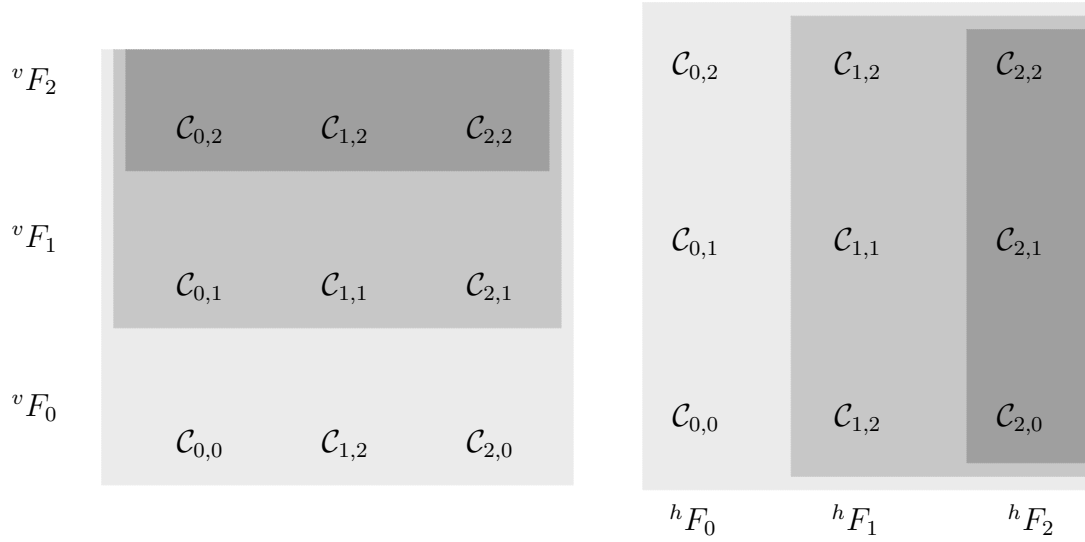


Figure 2.5

2.2.4 Spectral sequences from bicomplexes

The standard filtrations vF and hF induce spectral sequences of filtered complexes ${}^{hv}E$ and ${}^{vh}E$, respectively.

For example, consider the row-wise filtration vF shown on the left of Figure 2.5. The pages of the attached spectral sequence ${}^{hv}E$ approximate $H_*(Tot(\mathcal{C}))$ via the following process. On page ${}^{hv}E^0$, the differential d^0 consists of all the components of ∂^{Tot} which preserve the filtration grading; that is, $d^0 = \partial^h$. The induced differential on the next page $(\partial^{Tot})'$ now only has components which shift the filtration grading by 1 or more.

The following proposition says that in certain situations, both spectral sequences eventually collapse to ${}^{hv}E^\infty \cong {}^{vh}E^\infty \cong H_*(Tot(\mathcal{C}))$, where the vector space generated by the surviving basis elements at the level set with filtration level $p+q = m$ is identified with $H_m(Tot(\mathcal{C}))$. A proof may be found in [McC01].

Proposition 2.11. *If for each n , the number of p such that $\mathcal{C}_{p,n-p} \neq 0$ is finite, then*

$${}^{hv}E_{p,q}^\infty = {}^vF_q H_{p+q}(Tot(\mathcal{C})) / {}^vF_{q+1} H_{p+q}(Tot(\mathcal{C}))$$

$${}^{vh}E_{p,q}^\infty = {}^hF_p H_{p+q}(Tot(\mathcal{C})) / {}^hF_{p+1} H_{p+q}(Tot(\mathcal{C})).$$

Both spectral sequences converge to the total homology.

2.2.5 Equivariant homology and the Tate spectral sequence

Our bicomplex will be modeled on the bicomplex used to compute the Borel equivariant cohomology of a topological space X with an involution τ :

$$H_{\mathbb{Z}/2\mathbb{Z}}^*(X; \mathbb{F}_2) := \text{Ext}_{\mathbb{F}_2[\mathbb{Z}/2\mathbb{Z}]}(C_*(X), \mathbb{F}_2).$$

Here $C_*(X)$ is the singular chain complex for X with \mathbb{F}_2 coefficients. Since we are working over \mathbb{F}_2 coefficients, $(1 + \tau)^2 = 0$, so this is indeed a complex. Moreover, one can check that the differentials of $C_*(X)$ and the induced involution $\tau^\#$ commute, and therefore so do their duals. Thus we can build the double complex

$$0 \rightarrow C^*(X; \mathbb{F}_2) \xrightarrow{1+\tau^\#} C^*(X; \mathbb{F}_2) \xrightarrow{1+\tau^\#} C^*(X; \mathbb{F}_2) \xrightarrow{1+\tau^\#} \dots$$

which computes the Borel equivariant cohomology of X . Equipped with the row-wise and column-wise filtration, this is a bicomplex.

We can view the underlying vector space of the bicomplex as a module $C^*(X; \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta]$, where θ shifts the column-wise filtration degree by 1 (to the right).

Localizing at θ gives the bi-infinite *Tate bicomplex*

$$C^*(X; \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}] = \left(\dots \xrightarrow{1+\tau^\#} C^*(X; \mathbb{F}_2) \xrightarrow{1+\tau^\#} C^*(X; \mathbb{F}_2) \xrightarrow{1+\tau^\#} \dots \right)$$

whose total homology, under some finiteness conditions, is isomorphic to $H^*(X^{\text{fix}}; \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$. Here X^{fix} is the τ -invariant topological subspace of X . See Section 2 in [LT16] for more details on $\mathbb{Z}/2\mathbb{Z}$ -localization in Borel equivariant cohomology.

2.3 Topological preliminaries

The purpose of this section is to give an overview of the *annular Khovanov homology* of a link while establishing the notation to be used in the proof of the main result. In §2.3.1 we define annular links and relate them to braid and tangle closures. As annular Khovanov homology is computed from an annular link diagram, in §2.3.2 we set up the annular link diagram such that the topological involution on the link translates nicely to an involution on the diagram. §2.3.3 reviews the construction of the annular Khovanov

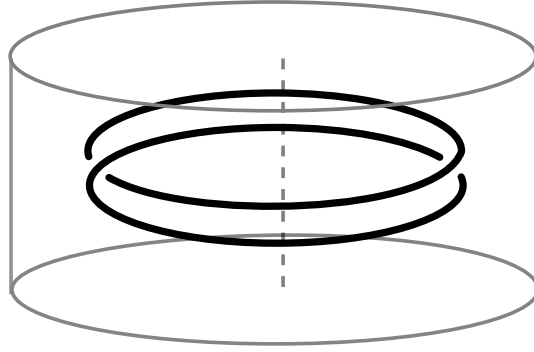


Figure 2.6: A Hopf link embedded in a thickened annulus.

chain complex via a *cube of resolutions* and defines the three gradings attached to the complex.

Throughout, we work in the smooth category with oriented links and \mathbb{F}_2 coefficients.

2.3.1 Annular links and tangles

An *annular link* is a finite disjoint union of embedded circles $\coprod_1^n S^1$ in a thickened annulus $A \times I$. See Figure 2.6 for an example.

Annular links are *tangle closures*. A *tangle* \mathcal{T} is a properly embedded compact 1-manifold in $D^2 \times I$, with $\partial\mathcal{T} \subset D^2 \times \{0, 1\}$. If the number of boundary components of \mathcal{T} on the two disks agree, then after arranging for $\partial\mathcal{T} = \{x_1, \dots, x_n\} \otimes \{0, 1\}$, we can glue $D^2 \times \{0\}$ to $D^2 \times \{1\}$ (via id_{D^2}) to obtain a link $\widehat{\mathcal{T}} = \mathcal{T}/(x_i, 0) \sim (x_i, 1)$ in a solid torus $D^2 \times I/(D^2 \times \{0\} \sim D^2 \times \{1\})$.

Braids constitute a well-studied case of tangles. Closures of n -strand braids form the braid group B_n . Isotopy classes of n -braid closures correspond to the conjugacy classes in B_n . With this important case in mind, we think of A as the xy -plane punctured at the origin $*$ and call $\{*\} \times I \subset \mathbb{R}^2 \times I \cong D^2 \times I$ the *braid axis*.

A generic projection of an annular link $L \subset A \times I$ to A is an immersion with at most double points, as in Figure 2.7.

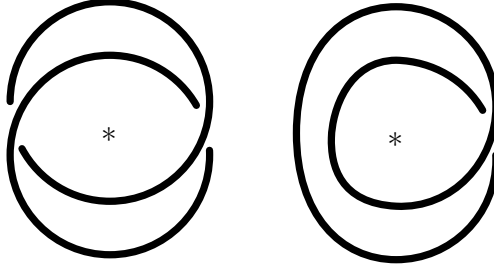


Figure 2.7: On the left is an annular diagram for the Hopf link from Figure 2.6. On the right is an annular diagram for the quotient knot, a positively stabilized annular unknot.

2.3.2 2-periodic links and their diagrams

Given a 2-periodic link $\tilde{L} \subset S^3$ with involution $\tau : S^3 \rightarrow S^3$, the fixed point set B (for “braid axis”) is an unknotted S^1 in S^3 .

With some choice of coordinates on S^3 minus a point $\{\infty\}$ on B , we can view B as the z -axis in \mathbb{R}^3 , τ as the rotation of 180° about the z -axis, and let A be the xy -plane minus the origin. Hence $\tilde{L} \hookrightarrow A \times \mathbb{R} \cong A \times I$ is an annular link.

Remark 2.12. We define a 2-periodic link $\tilde{L} \subset S^3$ as one that comes with a chosen involution of S^3 which fixes an unknot in S^3 . It is important to note that a link \tilde{L} in S^3 may come with multiple involutions τ_i , in the sense that the link $\tilde{L} \cup B_1$ is not isotopic to $\tilde{L} \cup B_2$ (where B_i is the fixed-point set of involution τ_i). In this case \tilde{L} with involution τ_1 defines a different annular link from \tilde{L} with involution τ_2 .

Maintaining symmetry under τ , isotope \tilde{L} so that the projection $\mathcal{D} : A \times I \rightarrow A$ takes \tilde{L} to a link diagram $\mathcal{D}(\tilde{L})$ in A . At the double points, we keep the crossing information. Employing a small isotopy, we may also assume the double points occur away from the y -axis, so that there is a clear notion of the “right side” of $A \times I$ or A (where $x \geq 0$), and the “left side” of $A \times I$ or A (where $x \leq 0$).

The preimage of the right side of $\mathcal{D}(\tilde{L})$ is a tangle \mathcal{T} , which by symmetry has the same number of loose strands on both ends. The quotient link of \tilde{L} with respect to the

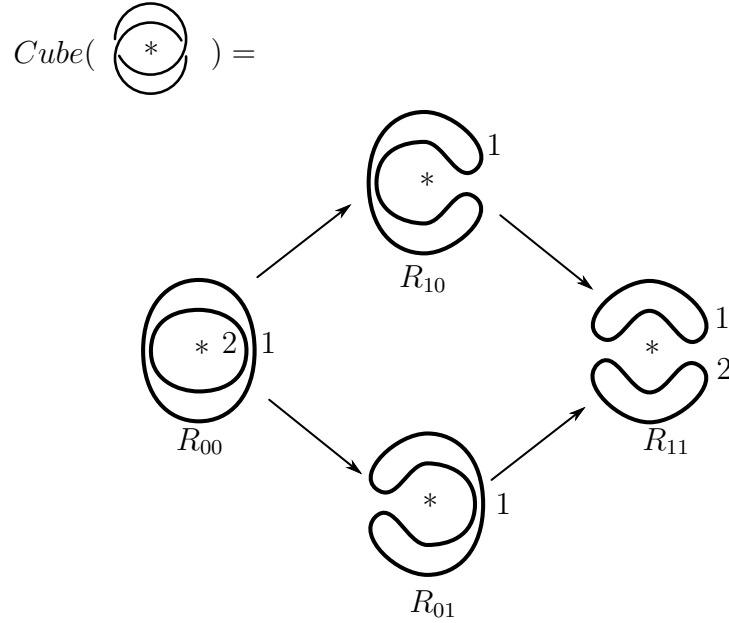


Figure 2.8: The cube of resolutions for the annular Hopf link in Figure 2.7.

involution τ is the tangle closure $L := \widehat{\mathcal{T}}$. A diagram for L in A is given by $\mathcal{D}(\mathcal{T})$ on the right side of A together with an identity braid on the left side of A . See Figure 2.7 for example.

2.3.3 Annular Khovanov homology

Annular Khovanov homology (with \mathbb{F}_2 coefficients) is a TQFT from properly embedded 1-manifolds in $A \times I$ and cobordisms between them to \mathbb{F}_2 vector spaces and linear maps between them. Asaeda, Przytycki, and Sikora define this annular link invariant in [APS04], where they construct a *cube of resolutions* from the diagram $\mathcal{D}(L)$ and associate to it a triply-graded chain complex $CKh(\mathcal{D}(L))$ which computes their annular link invariant $AKh(L)$.

We define two types of cubes, related by the TQFT. See Figure 2.8 and Figure 2.9 for examples.

Definition 2.13 (Cube of resolutions). We will use the notation $Cube(\mathcal{D})$ to denote the

$$\text{CUBE}\left(\begin{array}{c} \bigcirc \\ \bigcirc \\ * \end{array}\right) =$$

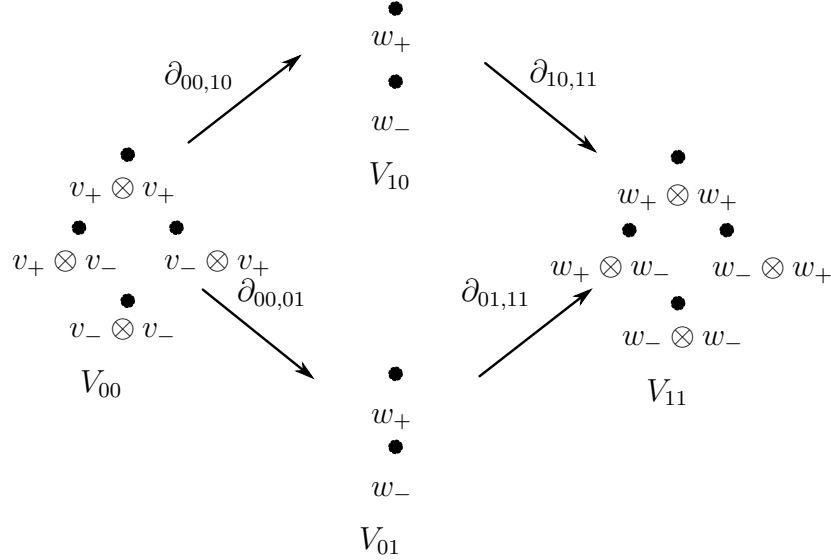


Figure 2.9: The cube of chains for the annular Hopf link in Figure 2.7.

cube of resolutions of the diagram \mathcal{D} , whose

- vertices are resolutions of \mathcal{D} , which are closed 1-manifolds embedded in A
- edges are 2-dimensional saddle cobordisms between two vertices.

Definition 2.14 (Cube of chains). The annular Khovanov TQFT takes $Cube(\mathcal{D})$ to $\text{CUBE}(\mathcal{D})$, the *cube of chains*, whose

- vertices are \mathbb{F}_2 vector spaces
- edges are linear maps between the vertices.

We outline the construction of $Cube(\mathcal{D})$ and $\text{CUBE}(\mathcal{D})$ in the rest of this subsection. Let n be the number of crossings in $\mathcal{D}(L)$, and label the crossings 1 through n in any order. Let n_+ denote the number of positive crossings, and n_- the number of negative crossings.

Definition 2.15 (Bitstring notation). We call $\alpha \in \{0, 1\}^n$ a *bitstring of length n* . There are many operations we can perform on α :

- $\alpha[i]$ denotes the bit at the i -th position, where indexing begins at $i = 0$.
- $|\alpha| = \sum_{i=0}^{n-1} \alpha[i]$ is the *Hamming weight* of α .
- Let $\alpha' \in \{0, 1\}^n$. Suppose there is an index k such that
 - $\alpha[k] = 0$
 - $\alpha'[k] = 1$
 - for all $j \neq k$, $\alpha[j] = \alpha'[j]$.

In this case, we say that α' is a *bit increment from α* .

- Let $\beta \in \{0, 1\}^n$. Bitstrings of length n form a poset: we write $\alpha < \beta$ if for all i , $\alpha[i] \leq \beta[i]$.
- $\alpha\beta$ represents the concatenation of α and β , in that order. That is, $\alpha\beta$ is the length $2n$ bitstring where

$$\alpha\beta[i] = \begin{cases} \alpha[i] & \text{for } i < n \\ \beta[i - n] & \text{for } i \geq n \end{cases}.$$

- For $\alpha\beta$ as above, there is an involution τ defined by $\tau(\alpha\beta) = \beta\alpha$.

Remark 2.16. We will also use the symbol “ τ ” to denote some topological involutions, but the distinction from this bitstring operation should be clear from context. See §2.4.1 for more on the notation for various involutions.

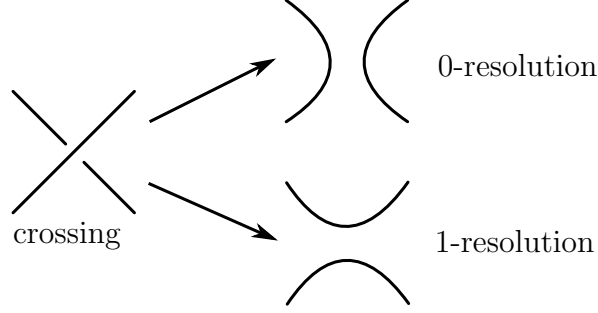


Figure 2.10

Vertices

Let $\alpha \in \{0, 1\}^n$ be a length n bitstring. Figure 2.10 shows two ways of resolving a local picture of a crossing. Associate to α a resolution R_α of $\mathcal{D}(L)$, where at crossing i we replace a local picture with a 0-resolution if $\alpha[i] = 0$, or a 1-resolution if $\alpha[i] = 1$. The resulting resolution R_α is a closed 1-manifold embedded in the annulus A .

Let $|R_\alpha|$ denote the number of components of R_α . Label the components $C_1, \dots, C_{|R_\alpha|}$. Let $\mathbb{V} = \langle v_-, v_+ \rangle$ be the 2-dimensional \mathbb{F}_2 vector space generated by the symbols v_- and v_+ . We associate to the resolution R_α a vector space $V_\alpha = \mathbb{V}^{\otimes |R_\alpha|}$, generated by the pure tensors of the form $v = v_{p_1} \otimes v_{p_2} \otimes \dots \otimes v_{p_{|R_\alpha|}} \in \{v_-, v_+\}^{\otimes |R_\alpha|}$. We endow $CKh(\mathcal{D}(L))$ with the following three gradings by assigning gradings to these distinguished generators. This notation is slightly modified from that in [GLW17].

- The (homological) i -grading of v is

$$gr_i(v) = |\alpha| - n_-.$$

- Define $p(v) = \#\{p_i : p_i = +\} - \#\{p_i : p_i = -\}$. The (quantum) j -grading of v is

$$gr_j(v) = p(v) + |\alpha| + n_+ - 2n_-.$$

- The components of R_α come in two flavors with respect to the basepoint $*$:
 - The *nontrivial circles* have nonzero winding number with respect to $*$ (with respect to either orientation).
 - The *trivial circles* have zero \mathfrak{sl}_2 weight space with respect to $*$ (with respect to either orientation).

The (\mathfrak{sl}_2 weight space) k -grading of v is given by

$$gr_k(v) = \#\{p_i : p_i = +, C_i \text{ nontrivial}\} - \#\{p_i : p_i = -, C_i \text{ nontrivial}\}.$$

We think of the i -th tensor factor in $\mathbb{V}^{\otimes |R_\alpha|}$ as corresponding to a labeling of the i -th component C_i with a $-$ or $+$ symbol. As such, for a trivial circle C_i we will henceforth write the corresponding factor as $\mathbb{W} = \langle w_-, w_+ \rangle$, reserving the notation $\mathbb{V} = \langle v_-, v_+ \rangle$ for nontrivial circles. For example, if R_α consists of one nontrivial circle and two trivial circles, labeled in that order, one of the generators of V_α is $v_- \otimes w_+ \otimes w_-$.

Remark 2.17. In [GLW18], Grigsby, Licata, and Wehrli show that annular Khovanov homology over \mathbb{C} enjoys a Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ -action, which is why we refer to the k -grading as the “ \mathfrak{sl}_2 weight space grading.”

Edges

Let $\alpha, \alpha' \in \{0, 1\}^n$. If α' is a bit increment from α , we say that the resolution R'_α is a *successor* to R_α . In our cube of resolutions, we draw a directed edge $R_\alpha \rightarrow R'_\alpha$ corresponding to a linear map $\partial_{\alpha, \alpha'} : V_\alpha \rightarrow V_{\alpha'}$. This map depends on a few characteristics of the resolutions R_α and $R_{\alpha'}$.

- Since R_α and $R_{\alpha'}$ differ at the site of exactly one crossing, all but one or two components remain unchanged between R_α and $R_{\alpha'}$. The linear map $\partial_{\alpha, \alpha'}$ may be

a *merge map* or a *split map*. If the site of the crossing touches only one component C_i of R_α , then in $R_{\alpha'}$, C_i has split into two components. If the site of the crossing touches two components C_i and C_j of R_α , then in $R_{\alpha'}$, C_i and C_j merge into one component.

- The participating components may be trivial or nontrivial circles. The linear map $\partial_{\alpha,\alpha'}$ is constructed so that gr_j and gr_k are preserved, and gr_i is increased by 1. Roberts shows in [Rob13] that there are exactly six possible types of maps corresponding to an edge in the cube resolutions. We can visually describe these maps by drawing dots representing the distinguished generators of V_α and $V_{\alpha'}$, aligning them in rows of constant gr_j , and drawing arrows between the dots to represent components of the differential. Figure 2.11 shows the six maps defining all possible $\partial_{\alpha,\alpha'}$ using this visual representation.

One can check that $\partial_{AKh} = \sum \partial_{\alpha,\alpha'}$ is a differential by verifying that each face of the cube of chains commutes. The triply graded homology of this complex $CKh(\mathcal{D}(L))$ is independent of the choice of diagram $\mathcal{D}(L)$ [APS04].

2.4 The annular Khovanov-Tate bicomplex

We are now prepared to construct the bicomplex which defines the spectral sequence in Theorem 2.1. The motivation for this construction is discussed in §2.2.5.

2.4.1 An involution on CKh

The first task is to define an involution on $CKh(\mathcal{D}(\tilde{L}))$ induced by the topological involution τ .

Definition 2.18. We will need to discuss many related involutions. By abuse of notation, we utilize only three symbols; the distinctions within each group will be clear from

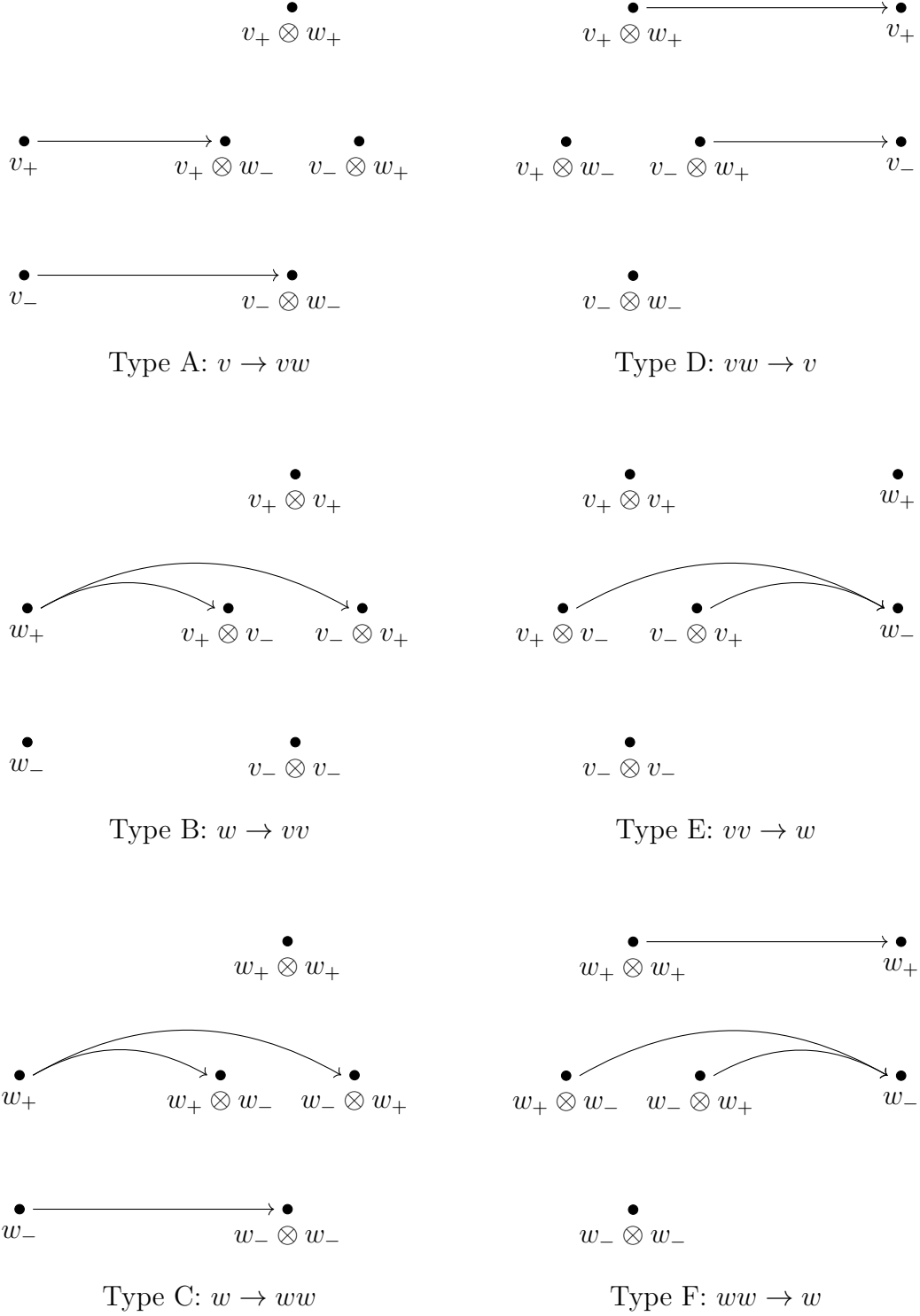


Figure 2.11: The AKh differentials. Within each diagram, the rows have constant j -grading.

context. After presenting the notation, we discuss the involutions and their relationships with each other in more detail.

In the following contexts, we use the symbol τ :

- the bitstring involution defined in Definition 2.15
- the topological involution on S^3 with fixed point set B
- the restriction of τ to the 2-periodic link $\tilde{L} \subset S^3$
- the involution induced on $A \times I$ and its restriction to $\tilde{L} \hookrightarrow A \times I$.

We use the symbol τ_A to denote the following:

- the involution induced on A from τ on $A \times I$
- the restriction of τ_A to $\mathcal{D}(\tilde{L}) \subset A$
- the involution induced on an equivariant resolution $R_{\alpha\alpha} \subset A$.

Finally, $\tau^\#$ is used to denote the following:

- the involution induced on the chains of $CKh(\mathcal{D}(\tilde{L}))$
- the involutions induced by $\tau^\#$ on the pages of the spectral sequences.

We continue to use the setting described in §2.3.2. Let n be the number of crossings in the quotient link diagram $\mathcal{D}(L)$. Label the $2n$ crossings in $\mathcal{D}(\tilde{L})$ by first labeling crossings of $\mathcal{D}(\tilde{L})$ on the right side of A ($x > 0$), then labeling the crossings on the left side ($x < 0$) so that τ_A takes the i -th crossing on the right to the $(i + n)$ -th crossing on the left.

With this assignment, each $2n$ -bit string $\alpha = \alpha_1\alpha_2$ is the concatenation of two n -bit strings α_1 and α_2 , where α_1 represents a sequence of resolution choices for the crossings

on the right and α_2 represents a sequence of resolution choices for the crossings on the left. Thus τ_A takes $R_\alpha = R_{\alpha_1\alpha_2}$ to $R_{\tau(\alpha)} = R_{\alpha_2\alpha_1}$.

Thinking of the distinguished generators of the cube of chains $\text{CUBE}(\mathcal{D}(\tilde{L}))$ as marked resolutions, τ_A induces an involution $\tau^\#$ on the chains of $CKh(\mathcal{D}(\tilde{L}))$.

Definition 2.19. A resolution R_α is *equivariant* if $\tau_A(R_\alpha) = R_\alpha$. A generator x of the cube of chains $\text{CUBE}(\mathcal{D}(\tilde{L}))$ is *equivariant* if $\tau^\#(x) = x$. If a resolution or generator is not equivariant, it is *nonequivariant*.

In fact, $\tau^\#$ is an involution on the complex $(CKh(\mathcal{D}(\tilde{L})), \partial_{AKh})$:

Lemma 2.20. ∂_{AKh} and $\tau^\#$ commute.

Proof. It suffices to show that every edge belongs to a well-defined pair of edges $\{\partial_{\alpha,\alpha'}, \partial_{\tau(\alpha),\tau(\alpha')}\}$ of $\text{CUBE}(\mathcal{D}(\tilde{L}))$, where α' is a bit increment from α , so that the following diagram commutes.

$$\begin{array}{ccc} V_{\alpha'} & \xrightarrow{\tau^\#} & \tau^\#(V_{\alpha'}) \\ \partial_{\alpha,\alpha'} \uparrow & & \uparrow \partial_{\tau(\alpha),\tau(\alpha')} \\ V_\alpha & \xrightarrow{\tau^\#} & \tau^\#(V_\alpha) \end{array}$$

Since ∂_{AKh} is the sum of all such pairs of edges, ∂_{AKh} commutes with $\tau^\#$.

The edge $\partial_{\alpha,\alpha'}$ may correspond to a change of resolution on the right or left side of $\mathcal{D}(\tilde{L})$. Without loss of generality, consider the case where the resolution change is on the right.

$$\begin{array}{ccc} V_{\alpha'_1\alpha_2} & \xrightarrow{\tau^\#} & V_{\alpha_2\alpha'_1} \\ \partial_{\alpha,\alpha'} \uparrow & & \uparrow \partial_{\tau(\alpha),\tau(\alpha')} \\ V_{\alpha_1\alpha_2} & \xrightarrow{\tau^\#} & V_{\alpha_2\alpha_1} \end{array}$$

The components of $R_{\alpha_1\alpha_2}$ and $R_{\alpha_2\alpha_1}$ are identified in pairs by the diffeomorphism τ_A ; the same holds for the pair $R_{\alpha'_1\alpha_2}$ and $R_{\alpha_2\alpha'_1}$. In this way, a merge (resp. split) map $V_{\alpha_1\alpha_2} \rightarrow V_{\alpha'_1\alpha_2}$ corresponds to the isomorphic merge (resp. split) map $V_{\alpha_2\alpha_1} \rightarrow V_{\alpha_2\alpha'_1}$.

Notice that $\partial_{\alpha, \alpha'} = \partial_{\tau(\alpha)}$ if and only if they have the same source and target. This is impossible as one of $|\alpha|$ and $|\alpha'|$ is odd, and hence corresponds to a nonequivariant resolution. \square

2.4.2 Construction of the $\mathrm{AKH}^{\mathrm{Tate}}$ bicomplex

This allows us to define a variant of the Tate bicomplex (see §2.2.5), constructed as follows:

- Each column is a copy of the complex $CKh(\mathcal{D}(\tilde{L}))$, where the vertical filtration is induced by gr_i .
- The underlying vector space is the $\mathbb{F}_2[\theta]$ -module $CKh(\mathcal{D}(\tilde{L})) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta]$, where θ acts on the columns by shifting right by one column.
- The horizontal differential is $1 + \tau^\#$.

Observe that this complex is quadruply graded by gr_i, gr_j, gr_k , and a column grading gr_t , where t is the exponent of θ .

Definition 2.21. We call this bicomplex the *annular Khovanov-Tate complex* for the 2-periodic link \tilde{L} with involution τ , abbreviated $\mathrm{AKH}^{\mathrm{Tate}}(\tilde{L})$. We denote the total complex of this bicomplex by $Tot(\mathrm{AKH}^{\mathrm{Tate}}(\tilde{L}))$.

Remark 2.22. This notation requires some justification. We will show that

$$H^*(Tot(\mathrm{AKH}^{\mathrm{Tate}}(\tilde{L}))) \cong AKh(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$$

where the latter is independent of the choice of diagram by [APS04].

2.5 Proof of Theorem 2.1

We are now ready to prove the main result. Recall that $AKh^{j,k}(L)$ denotes the part of $AKh(L)$ at quantum grading j and \mathfrak{sl}_2 weight space grading k .

Theorem 2.1. *Let \tilde{L} be a 2-periodic link with quotient link L . For each pair of integers (j, k) , there is a spectral sequence with*

$$E^1 \cong AKh^{2j-k,k}(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}] \Rightarrow E^\infty \cong AKh^{j,k}(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

Before diving into the details, we first give a sketch of the main ideas behind the proof. The reader may also wish to refer to the example in §2.6.

Since the complex $CKh(\mathcal{D}(\tilde{L}))$ is finite-dimensional, our bicomplex $AKH^{\text{Tate}}(\mathcal{D}(\tilde{L}))$ has finite-dimensional columns, so by Proposition 2.11, the ${}^{hv}E$ and ${}^{vh}E$ spectral sequences both converge to $H_*(Tot(AKH^{\text{Tate}}(\mathcal{D}(\tilde{L}))))$. We will compute the ${}^{hv}E$ spectral sequence to find that ${}^{hv}E^\infty \cong AKh(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$. Then we will show that ${}^{vh}E^1 = AKh(\tilde{L}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$. By Proposition 2.11, ${}^{vh}E^\infty \cong {}^{hv}E^\infty$, with the isomorphism respecting the $(i+t)$ -grading. Observing that each diagonal level set in ${}^{vh}E^\infty$ can be identified with $\oplus_i AKh_i(L)$ (where $AKh_i(L)$ refers to the piece of $AKh(L)$ at $gr_i = i$), we see that ${}^{vh}E$ is a spectral sequence from the $(i+t)$ -filtered $AKh(\tilde{L}) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$ to $AKh(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}]$. The total differential preserves j - and k -gradings, so ${}^{vh}E$ splits along j - and k -gradings.

2.5.1 The first pages

The first pages of either spectral sequence can be understood readily and have topological interpretations.

Lemma 2.23. ${}^{vh}E^1 \cong AKh(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$.

Proof. In each column, the cancellation of all the vertical arrows computes the homology $AKh(\mathcal{D}(\tilde{L}))$. The fact that cancellation actually occurs in the greater context of the bicomplex is irrelevant because any induced maps are no longer vertical. \square

Lemma 2.24. ${}^{hv}E^1$ is generated by the equivariant generators of $AKH^{Tate}(\mathcal{D}(\tilde{L}))$.

Proof. Since $1 + \tau^\#$ vanishes on all equivariant generators, it suffices to show that all the nonequivariant generators vanish upon cancellation of all the horizontal arrows in the bicomplex. In fact, cancellation of all the $\tau^\#$ arrows suffices, as shown in Figure 2.12. \square

Definition 2.25. A *staircase of arrows* (Figure 2.13) is a finite collection of arrows in a bicomplex (generated by a distinguished filtered graded basis) which form a path with the following properties.

- The path alternates between vertical and horizontal arrows.
- Both arrows at a corner either both point away from or both point toward the corner.
- If a vertical arrow and a horizontal arrow meet at a corner of the path, they either have the same source or the same target.

The *height* of a staircase of arrows with respect to a given filtration is the change in filtration degree between the generators at the ends of the path.

With respect to the row-wise filtration, if both ends of the staircase begin with a vertical arrow, then cancellation of the horizontal arrows in the staircase (in any order) induces a single arrow from one end of the path to the other. The *length* of this induced arrow is the height of the staircase.

Lemma 2.26. If r is odd, ${}^{hv}d^r = 0$.

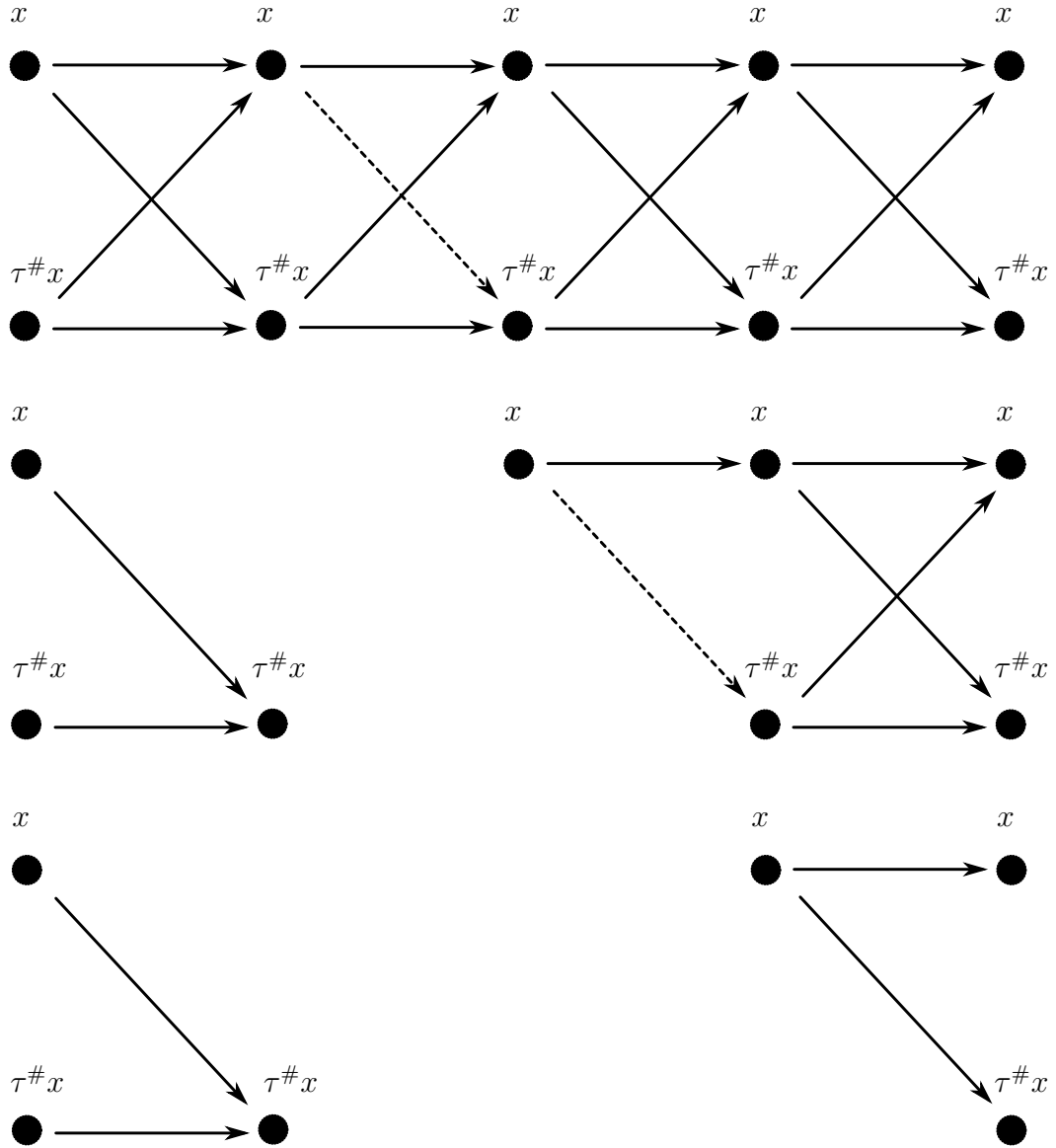


Figure 2.12: Cancellation of the $\tau^\#$ arrows eliminates all the nonequivariant generators.

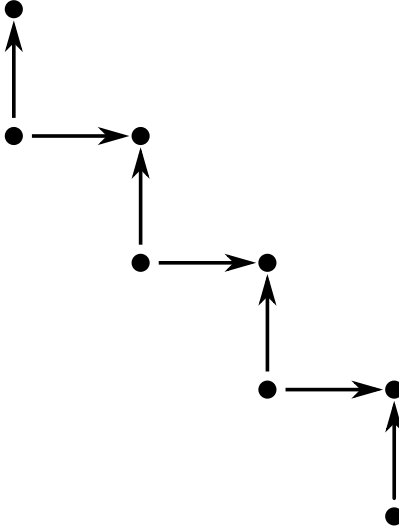


Figure 2.13: A staircase of arrows. In this example, the staircase has *height* 4, so it induces an arrow of *length* 4.

Proof. By Lemma 2.24, only equivariant generators survive to ${}^{hv}E^1$. Since these belong to vector spaces corresponding to equivariant resolutions of $\mathcal{D}(\tilde{L})$, their Hamming weights must be even. The i -grading is a shift of Hamming weight, so every other row vanishes on ${}^{hv}E^1$. Components of ${}^{hv}d_r$ for odd r are represented by arrows of length r , which must map to or from a vanishing row, so ${}^{hv}d_r = 0$. \square

Therefore the next interesting differential in the spectral sequence is ${}^{hv}d^2$.

Lemma 2.27. *Every component of ${}^{hv}d^2$ is induced by the cancellation of a $\tau^\#$ arrow at a row corresponding to an odd Hamming weight.*

Proof. Since we begin with only (row-wise) filtration degree 0 and 1 maps, the only way to induce a length 2 differential is by canceling the horizontal arrow on a staircase of height 2. The source and target of a length 2 differential is an equivariant generator, so the cancelled horizontal arrow must lie in a row corresponding to an odd Hamming weight. \square

Remark 2.28. Later on it will become useful to view the computation of the induced differentials in the spectral sequence as a sequence of maps (by ∂_{AKh}) and lifts (by

$1 + \tau^\#$), traveling up the staircase in Figure 2.13. The maps constructed this way are the same as the maps induced by cancellation of the $\tau^\#$ arrows in the staircase. This is a more intuitive reason why cancellation of only the $\tau^\#$ arrows is sufficient for eliminating the nonequivariant generators.

2.5.2 Relationship between $({}^{hv}E^2, {}^{hv}d^2)$ and the quotient link

The crux of the theorem comes from the observation that the generators and differentials on page 2 of the ${}^{hv}E$ spectral sequence correspond exactly with distinguished generators in the cube of chains for the diagram of the quotient link. The following two propositions verify the details of this correspondence.

Proposition 2.29. *There is a one-to-one correspondence*

$$\begin{aligned} \{\text{generators of } {}^{hv}E^2\} &\longleftrightarrow \{\text{generators of } CKh(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]\} \\ \tilde{v} &\longleftrightarrow v \end{aligned}$$

induced by τ such that

- $gr_i(\tilde{v}) = 2gr_i(v)$
- $gr_j(\tilde{v}) = 2gr_j(v) - gr_k(v)$
- $gr_k(\tilde{v}) = gr_k(v)$
- $gr_t(\tilde{v}) = gr_t(v)$.

Proof. By Lemma 2.24, ${}^{hv}E^1$ is generated by the equivariant generators of $AKH^{\text{Tate}}(\mathcal{D}(\tilde{L}))$, and Lemma 2.26 shows ${}^{hv}d^1 = 0$, so ${}^{hv}E^1 = {}^{hv}E^2$.

Bijection of generators

First of all, the correspondence is given by the identity on the $\mathbb{F}_2[\theta, \theta^{-1}]$ factor.

Let \tilde{v} denote a generator of ${}^{hv}E^2$, coming from the equivariant resolution $R_{\alpha\alpha}$. Thinking of \tilde{v} as a τ_A -equivariant assignment of plusses and minuses to the components of $R_{\alpha\alpha}$, let v be the quotient of \tilde{v} by τ_A .

In the other direction, for v a generator of $CKh(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$ coming from a resolution R_α , view v as an assignment of plusses and minuses to the components of R_α . Pick any path connecting the two components of ∂A and take a double cover of R_α . The lift of v is \tilde{v} .

Grading relationships

The t -grading relationship follows by definition. For the other three relationships, suppose v consists of

- a nontrivial circles labeled “+”
- b nontrivial circles labeled “−”
- c trivial circles labeled “+”
- d trivial circles labeled “−”

In this case, write $v = v_+^a v_-^b w_+^c w_-^d$ to denote the distinguished generator, suppressing the tensor signs in favor of compact notation. Then $\tilde{v} = v_+^a v_-^b w_+^{2c} w_-^{2d}$.

Let n_+ and n_- denote the number of positive and negative crossings in $\mathcal{D}(L)$, respectively. For v ,

- $gr_i(v) = |\alpha| - n_-$
- $gr_j(v) = |\alpha| + (a - b + c - d) + n_+ - 2n_-$

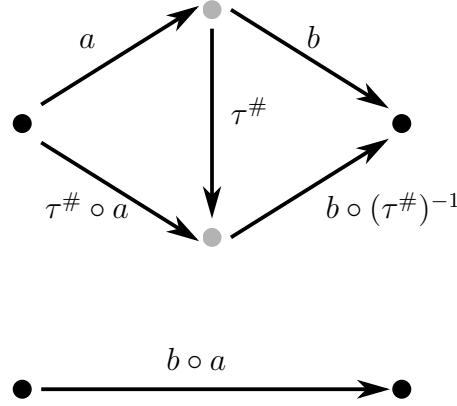


Figure 2.14: Cancellation of $\tau^\#$ arrows induce ${}^{hv}d^2$ arrows. For each diamond of the form above, we obtain a length 2 differential.

- $gr_k(v) = a - b$.

Since the diagram $\mathcal{D}(\tilde{L})$ obtained by taking the double cover of $\mathcal{D}(L)$ has twice as many positive and negative crossings as $\mathcal{D}(L)$, respectively, we compute that

- $gr_i(\tilde{v}) = 2|\alpha| - 2n_- = 2gr_i(v)$
- $gr_j(\tilde{v}) = 2|\alpha| + (a - b + 2c - 2d) + 2n_+ - 4n_- = 2gr_j(v) - (a - b) = 2gr_j(v) - gr_k(v)$
- $gr_k(\tilde{v}) = a - b = gr_k(v)$.

□

Proposition 2.30. *Under the correspondence above, each slope -2 line in the bicomplex $({}^{hv}E^2, {}^{hv}d^2)$ is isomorphic to the complex $(CKh(L), \partial_{AKh})$.*

Proof. Let α' be a bit increment from α . Lemma 2.27 implies that any component of ${}^{hv}d^2$ lies along a slope -2 line.

Upstairs, consider the components of ${}^{hv}d^2$ from $V_{\alpha\alpha} \otimes \theta^t$ to $V_{\alpha'\alpha'} \otimes \theta^{t-1}$. (See Figure 2.14.) The two resolutions $R_{\alpha\alpha'}$ and $R_{\alpha'\alpha}$ connecting $R_{\alpha\alpha}$ and $R_{\alpha'\alpha'}$ correspond to changing the resolutions of some pair of crossings c_i and c_{i+n} from their 0-resolution in $R_{\alpha\alpha}$ to their 1-resolution in $R_{\alpha'\alpha'}$.

Downstairs, consider the edge of $\text{CUBE}(\mathcal{D}(L))$ going from V_α to $V_{\alpha'}$. This edge falls into one of the six types of AKh differentials, as discussed in Section 2.3.3.

We compare the $^{hv}d^2$ components induced by the cancellation of the $\tau^\#$ arrows going between $V_{\alpha\alpha'}$ and $V_{\alpha'\alpha}$ with the components of ∂_{AKh} , in the six different cases. Figures 2.15 through 2.19 show the computation of $^{hv}d^2$ in all six cases. Compare these figures with Figure 2.11. Note that in contrast with the notation used in Figure 2.11, the tensor signs are now suppressed for the sake of compactness. By inspection, under the correspondence from Proposition 2.29, the induced length 2 arrows upstairs align exactly with the differential downstairs.

□

2.5.3 Visualizing higher differentials

Just as all components of $^{hv}d^2$ are induced by cancellation of some $\tau^\#$ arrow, longer differentials are induced by the cancellation of multiple $\tau^\#$ arrows. This can be seen by inducting on the applications of the cancellation operation that result in a differential of length $2r$. At each step, the slope of the arrows representing the induced differentials are each of the form $-\frac{k}{k-1}$. The generators at the ends of the staircase must be equivariant, and all the intermediate generators (i.e. the corners of the staircase) must be nonequivariant because they are adjacent to $\tau^\#$ arrows.

To prove that there are no higher differentials in the ^{hv}E spectral sequence, we focus on individual staircases. This directs us to restrict our focus to a local piece of the complex surrounding the staircase which is large enough to determine the behavior of the map induced by the staircase.

Definition 2.31. Let \mathcal{D} be an annular link diagram with n crossings. Let α and β be two n -bit strings where $\alpha < \beta$. The *subcube of resolutions* from R_α to R_β , denoted $\text{Cube}(R_\alpha, R_\beta)$, is the subgraph of $\text{Cube}(\mathcal{D})$ whose

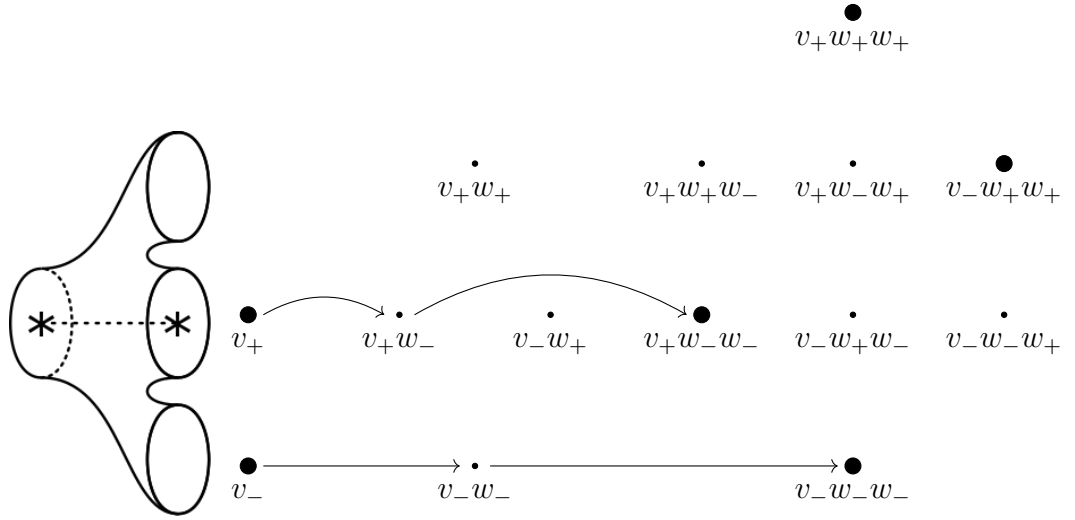


Figure 2.15: Type \mathcal{A} : $\tilde{v} \rightarrow \tilde{v}\tilde{w}$

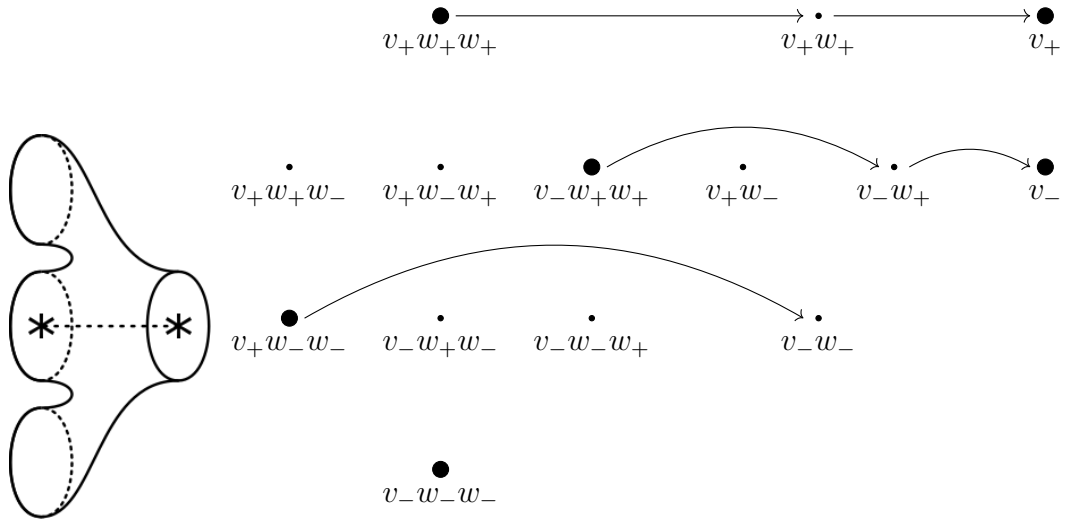
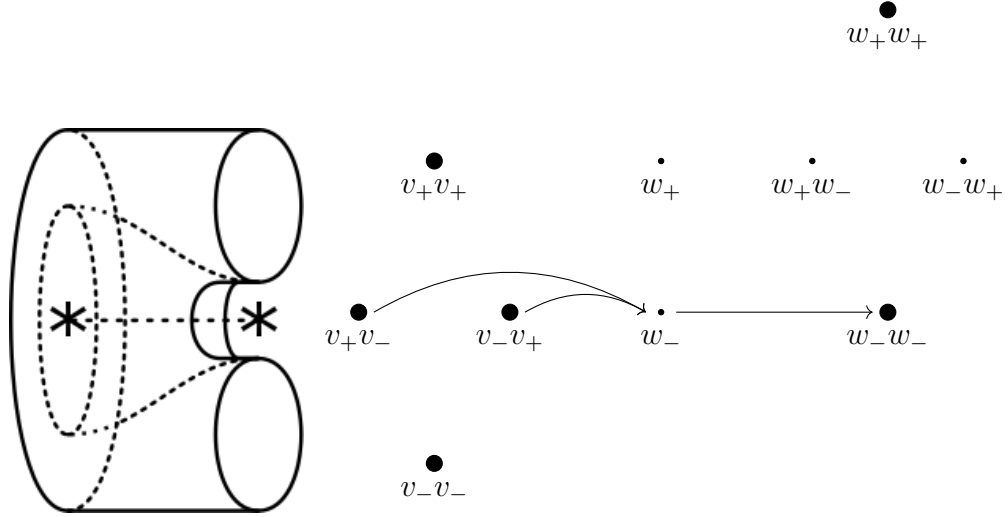
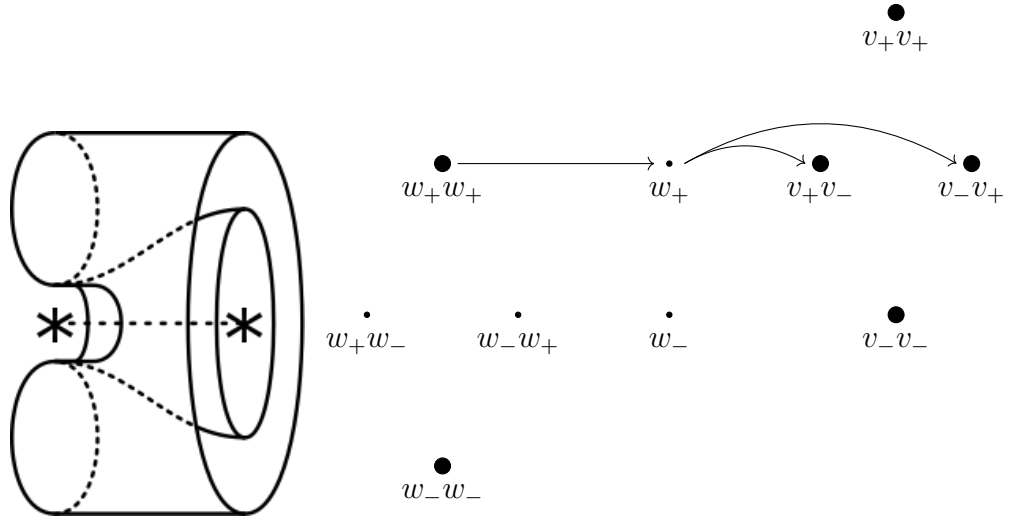


Figure 2.16: Type \mathcal{B} : $\tilde{v}\tilde{w} \rightarrow \tilde{v}$



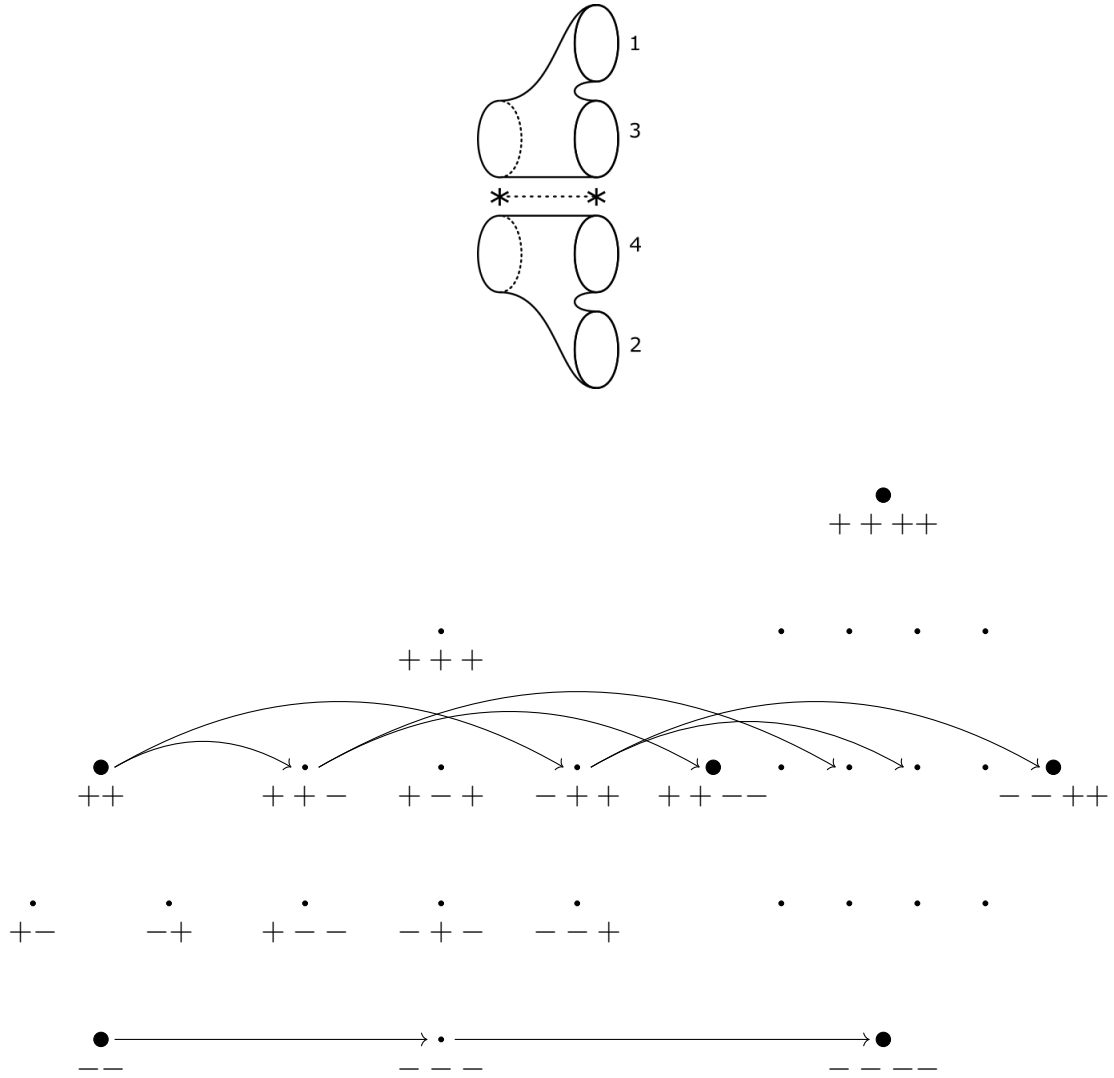


Figure 2.19: Type \mathcal{C} : $\tilde{w} \rightarrow \tilde{w}\tilde{w}$. We begin with $C_1 \cup C_2$. In the first step, C_1 splits into $C_1 \cup C_3$. In the second step, C_2 splits into $C_2 \cup C_4$.

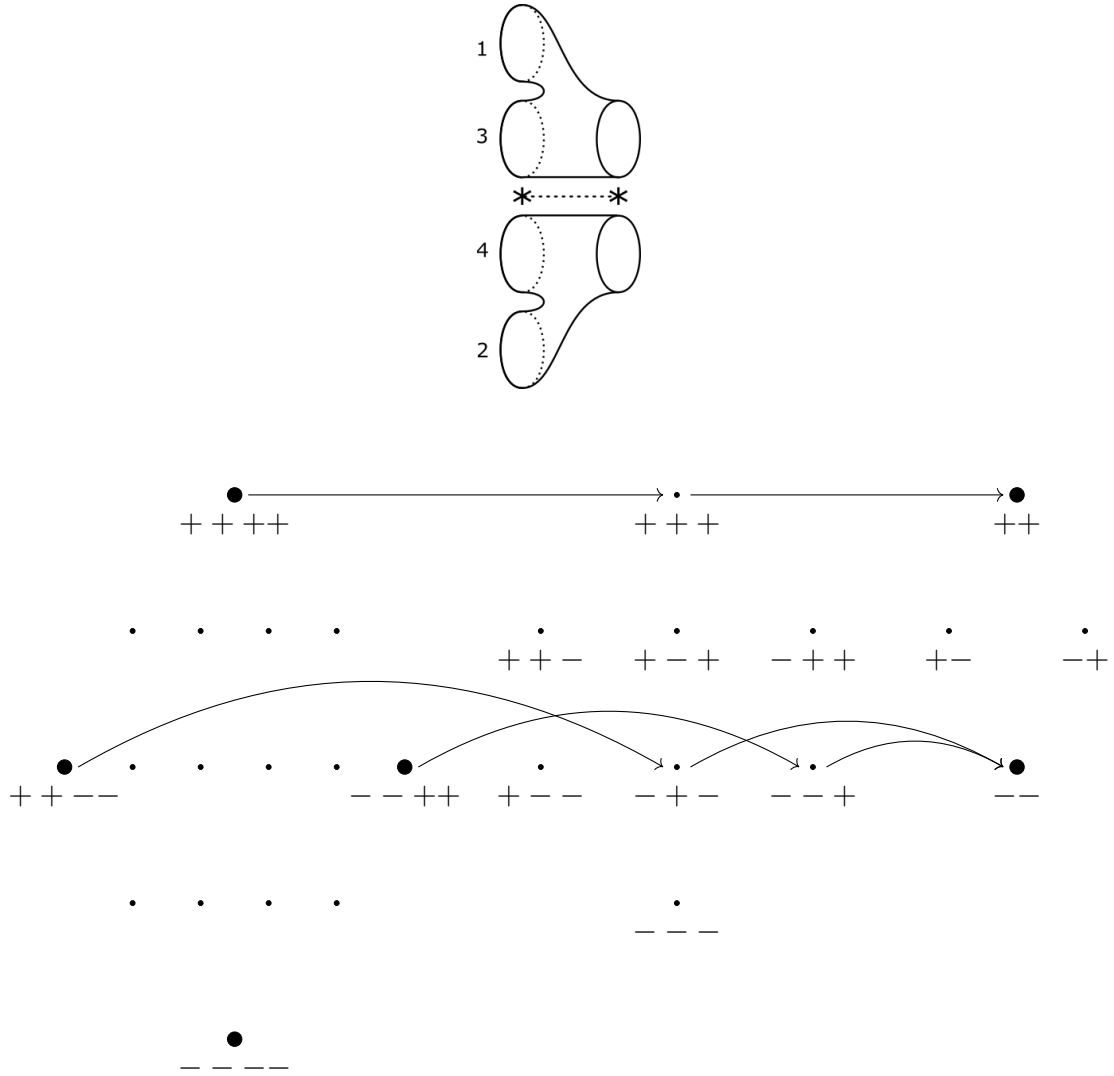


Figure 2.20: Type F: $\tilde{w}\tilde{w} \rightarrow \tilde{w}$. We start with $C_1 \cup C_2 \cup C_3 \cup C_4$, where $C_1 \cup C_2$ and $C_3 \cup C_4$ are each equivariant pairs of circles. In the first step, $C_1 \cup C_3$ merges to C_1 and C_2 stays put. In the second step, $C_2 \cup C_4$ merges to C_2 and C_1 stays put.

- vertex set consists of resolutions R_γ , where γ is an n -bit string with $\gamma[i] = \alpha[i] = \beta[i]$ whenever $\alpha[i] = \beta[i]$
- edge set consists of all edges of $Cube(\mathcal{D})$ whose endpoints are both in the vertex set.

In $CUBE(\mathcal{D})$, the *subcube of chains* from V_α to V_β , denoted $CUBE(V_\alpha, V_\beta)$, is defined similarly, with

- underlying chain groups consisting of the vector spaces corresponding to the vertex set of $Cube(R_\alpha, R_\beta)$
- differentials corresponding to the edge set of $Cube(R_\alpha, R_\beta)$.

In this case the crossings c_i in the diagram \mathcal{D} for which $\alpha[i] \neq \beta[i]$ are called *participating crossings*, and all other crossings are called *nonparticipating crossings*. Let $\mathcal{D}(\alpha, \beta)$ denote the link diagram obtained from $\mathcal{D}(L)$ by smoothing only the nonparticipating crossings based on the common values of α and β at these bits. Then $Cube(\mathcal{D}(\alpha, \beta)) \cong Cube(R_\alpha, R_\beta)$ and $CUBE(\mathcal{D}(\alpha, \beta)) \cong CUBE(V_\alpha, V_\beta)$.

Similarly, the components of R_α fall into two categories: the *participating circles*, which are adjacent to the sites of participating crossings, and the *nonparticipating circles*, which are the components of R_α that do not participate in a merge or split along any edge in $Cube(R_\alpha, R_\beta)$.

Definition 2.32. If R_α and R_β are equivariant resolutions of an equivariant annular link diagram \mathcal{D} , and $\alpha < \beta$, $Cube(R_\alpha, R_\beta)$ is *equivariantly split* if the set of participating circles can be divided into two or more equivalence classes under the equivalence relation

$$C_i \sim C_j \text{ if there is a (site of a) crossing adjacent to both } C_i \text{ and } C_j.$$

Proposition 2.33. *For an equivariantly split 2-periodic diagram \mathcal{D} with $2r$ crossings, there are no induced differentials of length $2r$.*

Proof. If \mathcal{D} were equivariantly split, it could be written as a split diagram $\mathcal{D}_1 \coprod \mathcal{D}_2$, and hence

$$\text{CUBE}(\mathcal{D}) \cong \text{CUBE}(\mathcal{D}_1) \otimes \text{CUBE}(\mathcal{D}_2).$$

Let $\emptyset \neq \{C_j\}_{j \in J_1}$ denote the set of circles in R_α corresponding to the all-zeros resolution of \mathcal{D}_1 , and $\emptyset \neq \{C_j\}_{j \in J_2}$ the set of circles in R_α corresponding to the all-zeros resolution of \mathcal{D}_2 .

The marked resolution x corresponds to a choice of markings $a \in \{v_-, v_+\}^{|J_1|}$ for $\{C_j\}_{j \in J_1}$ plus a choice of markings $\{v_-, v_+\}^{|J_2|}$ for $\{C_j\}_{j \in J_2}$, so for computational reasons we write $x = a \otimes b$, even though in reality we might have ordered the tensor components differently. (We are implicitly using the isomorphism $\text{CUBE}(\mathcal{D}) \cong \text{CUBE}(\mathcal{D}_1) \otimes \text{CUBE}(\mathcal{D}_2)$.) Since $x = a \otimes b$ is an equivariant generator, a and b must be equivariant generators for $\text{CUBE}(\mathcal{D}_1)$ and $\text{CUBE}(\mathcal{D}_2)$, respectively.

As the differential respects the tensor factors, we may use the product rule to compute $d^{2r}(x)$.

$$\begin{aligned} x &= a \otimes b \\ \partial x &= \partial a \otimes b + a \otimes \partial b \\ (1 + \tau^\#)^{-1} \partial x &= (1 + \tau^\#)^{-1} \partial a \otimes b + a \otimes (1 + \tau^\#)^{-1} \partial b \\ \partial(1 + \tau^\#)^{-1} \partial x &= \partial(1 + \tau^\#)^{-1} \partial a \otimes b + (1 + \tau^\#)^{-1} \partial a \otimes \partial b + \partial a \otimes (1 + \tau^\#)^{-1} \partial b \\ &\quad + a \otimes \partial(1 + \tau^\#)^{-1} \partial b \\ &\dots \end{aligned}$$

Here we abuse notation and write ∂ to mean the differential in $\text{CUBE}(\mathcal{D})$, $\text{CUBE}(\mathcal{D}_1)$, or $\text{CUBE}(\mathcal{D}_2)$, depending on context. Writing $\tilde{\partial} := (1 + \tau^\#)^{-1} \partial$, where $(1 + \tau^\#)^{-1}$ represents

“lifting by $1 + \tau^\#$ ” (see Remark 2.28), the above computation becomes

$$\begin{aligned}
d^{2r}x &= \partial \tilde{\partial}^{2r-1}x \\
&= \partial \sum_{i=0}^{2r-1} \binom{2r-1}{i} \tilde{\partial}^i a \otimes \tilde{\partial}^{2r-1-i} b \\
&= \sum_{i=0}^{2r-1} \binom{2r-1}{i} \partial \tilde{\partial}^i a \otimes \tilde{\partial}^{2r-1-i} b + \tilde{\partial}^i a \otimes \partial \tilde{\partial}^{2r-1-i} b.
\end{aligned}$$

We want to show that each summand must be 0 on E^{2r} , where all the survivors are necessarily equivariant. Note that in order for $a' \otimes b' \in \text{CUBE}(\mathcal{D}_1) \otimes \text{CUBE}(\mathcal{D}_2)$ to be equivariant, a' and b' must be simultaneously equivariant, and hence must fall in a column corresponding to even Hamming weight in their respective CUBES.

Case i odd

Then $\partial \tilde{\partial}^i a$ and $\tilde{\partial}^{2r-1-i} b$ have even Hamming weight, while $\tilde{\partial}^i a$ and $\partial \tilde{\partial}^{2r-1-i} b$ have odd Hamming weight. If any component of $\partial \tilde{\partial}^i a \otimes \tilde{\partial}^{2r-1-i} b$ is equivariant, then $\tilde{\partial}^{2r-1-i} b$ is equivariant, which contradicts the fact that $(1 + \tau^\#) \tilde{\partial}^{2r-1-i} b = \partial \tilde{\partial}^{2r-2-i} b \neq 0$.

Case i even

Then $\partial \tilde{\partial}^i a$ and $\tilde{\partial}^{2r-1-i} b$ have odd Hamming weight, while $\tilde{\partial}^i a$ and $\partial \tilde{\partial}^{2r-1-i} b$ have even Hamming weight. If any component of $\tilde{\partial}^i a \otimes \partial \tilde{\partial}^{2r-1-i} b$ is equivariant, then $\tilde{\partial}^i a$ is equivariant, contradicting the fact that $(1 + \tau^\#) \tilde{\partial}^i a = \partial \tilde{\partial}^{i-1} a \neq 0$.

□

2.5.4 Grading shifts of higher differentials

Again let α, β be equivariant $2r$ -bit strings with $\alpha < \beta$. By Proposition 2.33, we may now assume that $\mathcal{D}(\alpha, \beta)$ is not equivariantly split. This will allow us to perform computations by describing parts of the complex using connected graphs.

A first obstruction to the existence of a ${}^{hv}d^{2r}$ map between distinguished generators x and y is that x and y must have the same j -grading. For this reason, we compare the j -span of V_α and V_β , defined as follows.

Definition 2.34. The j -span of the vector space V_ω associated with the binary string ω is the range of j -gradings of the distinguished generators of V_ω .

Split maps shift the upper bound of the j -span up by 2, and merge maps shift the lower bound of the j -span up by 2.

Definition 2.35. A *path* through a cube of resolutions is a sequence of vertices R_{α_i} such that α_{i+1} is a bit increment from α_i , together with the edges between successive resolutions.

Since any component of ${}^{hv}d^{2r}$ will come from repeatedly mapping via ∂ and lifting by $1 + \tau^\#$ along a (directed) path from V_α to V_β in $\text{CUBE}(\mathcal{D}(\alpha, \beta))$, and $1 + \tau^\#$ affects neither of the j - or k -gradings, we focus on paths through $\text{CUBE}(\mathcal{D}(\alpha, \beta))$ and which combinations of merges and splits they involve.

Lemma 2.36. *Any path through a cube $\text{Cube}(\mathcal{D})$ from the all-zeros resolution to the all-ones resolution have the same number of merges and splits (separately).*

Proof. Let $\#m$ and $\#\Delta$ denote the number of merge and split maps along a path from V_α to V_β . These quantities satisfy the linear equations

$$\#m + \#\Delta = 2r$$

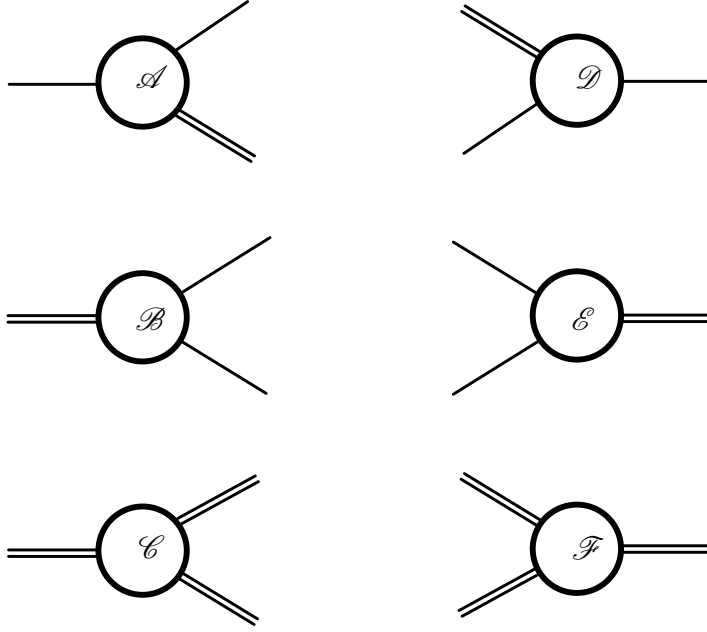


Figure 2.21: The six *atoms* representing the six possible equivariant two-steps.

$$|R_\beta| - |R_\alpha| = \#\Delta - \#m.$$

□

So, in order to count merge and splits, we may without loss of generality choose any path we'd like. Choose γ to be a path through the cube for which every other vertex is equivariant. In other words, after every two steps along this path, we land on an equivariant resolution. Each such set of two-steps corresponds to one step in a path in $\text{CUBE}(\mathcal{D}(\alpha, \beta)/\tau_A)$, since each equivariant pair of crossings corresponds to a crossing downstairs. We can thus classify the six different two-step paths in γ by looking at the classification of edges in the cube downstairs.

Definition 2.37. We can describe each two-step using modular pieces of a graph, which we call *atoms* (Figure 2.21). These pieces are oriented in the sense that the lines on the left of the circle represent *input circles*, or circles in the source resolution, and the lines on the right of the circle represent the *output circles*, or circles in the target resolution of the two-step. A double bond indicates an equivariant twin pair of W circles, and a single

bond indicates a single equivariant V circle. While attaching these atoms together, we can only attach V circles to V circles, and equivariant pairs of W circles to equivariant pairs of W circles. A connected graph obtained by attaching these atoms together is called a *molecule*.

Since the $\mathcal{D}(\alpha, \beta)/\tau_A$ diagram downstairs is nonsplit, when the pieces of γ are attached at their equivariant ends, we obtain a single molecule Γ . Let A, B, C, D, E, F denote the number of modular pieces of type $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$, and \mathcal{F} respectively. In particular, this means that $A + B + C + D + E + F = r$.

Proposition 2.38. *Given an equivariant path γ through an equivariantly nonsplit cube, the size of the j -span overlap between the vector spaces corresponding to the all-zeros resolution V_α and the all-ones resolution V_β is*

$$2(A + B + 2C + D + E + 2F - N_b)$$

where

- A, B, C, D, E , and F is the number of atoms of type $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$, and \mathcal{F} , respectively, in the molecule Γ corresponding to γ
- N_b is the total number of bonds (counted with multiplicity) in Γ .

Proof. Let j_0 denote the minimal j -grading of a distinguished generator of V_α . The number of components of R_α is equal to the number of input circles of the atoms separately, minus the number of input circles which are then identified with output circles of other atoms, which is equal to the number of total bonds, counted with multiplicity. Let N_b denote the total number of bonds. The source j -span is therefore

$$[j_0, j_0 + 2(A + 2B + 2C + 3D + 2E + 4F - N_b)].$$

Traveling through the path γ , we ultimately shift the bottom of the j -span by (two times) the number of merges, which is $2(B + 2D + E + 2F)$. Hence the possible j -span overlap between V_α and V_β is

$$[j_0 + 2(B + 2D + E + 2F), j_0 + 2(A + 2B + 2C + 3D + 2E + 4F - N_b)]$$

which is a difference of $2(A + B + 2C + D + E + 2F - N_b)$.

□

Thus the largest j -span overlap using the atoms in Γ is achieved when Γ has a minimal total number of bonds.

Lemma 2.39. *The minimal total number of bonds achievable by a connected molecule Γ comprised of a fixed set of atoms is achieved when Γ is composed to two trees, one containing all the \mathcal{A} , \mathcal{B} , \mathcal{D} , and \mathcal{E} atoms, and the other containing all the \mathcal{C} and \mathcal{F} atoms, with the two trees connected by a unique double bond.*

Proof. The structure of a minimally-bonded Γ must be a tree: if Γ has a cycle, we can decrease the number of bonds by cutting any side of the cycle.

Furthermore, when attaching atoms of type \mathcal{A} , \mathcal{B} , \mathcal{D} , and \mathcal{E} , the number of bonds is minimized by connecting to the molecule via a single rather than a double bond. When these atoms are arranged as a subtree of Γ , the number of single bonds is minimized. The \mathcal{C} and \mathcal{F} atoms will invariably attach to the tree via a double bond, so the two-tree molecule described has the same number of double bonds as any minimally-bonded Γ .

□

2.5.5 Higher differentials vanish

The final major step in the proof of Theorem 2.1 uses the j - and k - gradings to show that the ${}^{hv}E$ spectral sequence collapses on ${}^{hv}E^3$.

Proposition 2.40. ${}^{hv}d^{2r} = 0$ for all $r \geq 2$.

Proof. By Proposition 2.33, we may assume that $Cube(R_\alpha, R_\beta)$ is equivariantly non-split. Pick an equivariant path γ through $Cube(R_\alpha, R_\beta)$, and let Γ denote the associated molecule. Let's first assume that at least one of the atoms of Γ is *not* of type \mathcal{C} or \mathcal{F} . In the worst case, the total number of bonds is the sum of $A + B + D + E - 1$ from the $\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{E}$ tree, $2C + 2F - 2$ from the \mathcal{C}, \mathcal{F} tree, and 2 from connecting the two trees. Hence $N_b = A + B + 2C + D + E + 2F - 1$. The size of the overlap is then

$$2(A + 2B + 2C + 3D + 2E + 4F - (A + B + 2C + D + E + 2F - 1)) - 2(B + 2D + E + 2F) = 2.$$

Therefore x must have either the maximum or next-to-maximum j -grading of the distinguished generators of V_α . Since x is equivariant, any equivariant pair of trivial circles must have the same marking, so since x can have at most one component marked with a v_- , all trivial circles in x are marked w_+ . Moreover, from Figures 2.15 through 2.20 we see that if any equivariant paths out of V_α are not of type \mathcal{E} , then x is not in $\ker({}^{hv}d^2)$. So all equivariant paths out of V_α must be of type \mathcal{E} . This indicates that x is a marked resolution consisting solely of v_+ -marked concentric circles. But this means x is unique in its k -grading in the entire $CUBE(V_\alpha, V_\beta)$. Therefore there cannot be a ${}^{hv}d^{2r}$ component from x to y in $CUBE(V_\alpha, V_\beta)$.

In the case where *all* the atoms are of type \mathcal{C} or \mathcal{F} , the worst case Γ has $2C + 2F - 2$ bonds, producing a j -span overlap of

$$2(2C + 4F - (2C + 2F - 2)) - 2(2F) = 4.$$

Therefore x must have at most two circles marked with w_- . Again, we see in Figures 2.19 and 2.20 that x is not in $\ker({}^{hv}d^2)$.

Hence there cannot be a ${}^{hv}d^{2r}$ component from (a θ -power translate of) x to (a

θ -power translate of) y . □

As $\mathrm{AKH}^{\mathrm{Tate}}(\mathcal{D}(\tilde{L}))$ is 1-periodic in the horizontal direction, its homology is 1-periodic as well, so the total rank along each diagonal corresponding to the induced filtration gradings on H^{Tot} is the same as the total rank down along a vertical column. Therefore $\mathrm{AKh}(L) \otimes_{\mathbb{F}_2} \mathbb{F}[\theta, \theta^{-1}] \cong {}^{hv}E^3 = {}^{hv}E^\infty \cong {}^{vh}E^\infty$ so the ${}^{vh}E$ spectral sequence has ${}^{vh}E^1 \cong \mathrm{AKh}(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}[\theta, \theta^{-1}]$ and ${}^{vh}E^\infty \cong \mathrm{AKh}(L) \otimes_{\mathbb{F}_2} \mathbb{F}[\theta, \theta^{-1}]$.

Since all the maps in the double complex preserve j - and k -grading, the spectral sequence splits along j - and k -gradings. This concludes the proof of Theorem 2.1.

Finally, since each page of a spectral sequence is a subquotient of the previous page, Corollary 2.2 follows:

Corollary 2.2. *For any annular tangle closure L , quantum grading j and \mathfrak{sl}_2 weight space grading k ,*

$$rk_{\mathbb{F}_2} \mathrm{AKh}^{j,k}(L) \leq rk_{\mathbb{F}_2} \mathrm{AKh}^{2j-k,k}(\tilde{L}).$$

2.6 Example: Hopf link and stabilized unknot

To help convey the construction to the reader, we describe the simplest nontrivial example, a two-crossing annular Hopf link as the 2-braid closure $\tilde{L} := \widehat{\sigma_1^2}$. The quotient link is a positively stabilized unknot $L = \widehat{\sigma_1}$.

Annular diagrams for both \tilde{L} and L are shown in Figure 2.7. We label the crossings of $\mathcal{D}(\tilde{L})$ in accordance to our convention (§2.4.1); the first crossing is on the right, and the second crossing is on the left. The cube of resolutions $\mathrm{Cube}(\mathcal{D}(\tilde{L}))$ is shown in Figure 2.8. In order to simplify the notation in this small example, we give the distinguished generators unique, concise names in the cube of chains shown in Figure 2.22. For example, “ a_{+-} ” is the generator $v_+ \otimes v_-$ in V_{00} , corresponding to the marked resolution shown in Figure 2.23.

$$\text{CUBE}\left(\begin{array}{c} \text{---} \\ \bigcirc \text{---} \\ \bigcirc \text{---} \\ \bigcirc \text{---} \end{array}\right) =$$

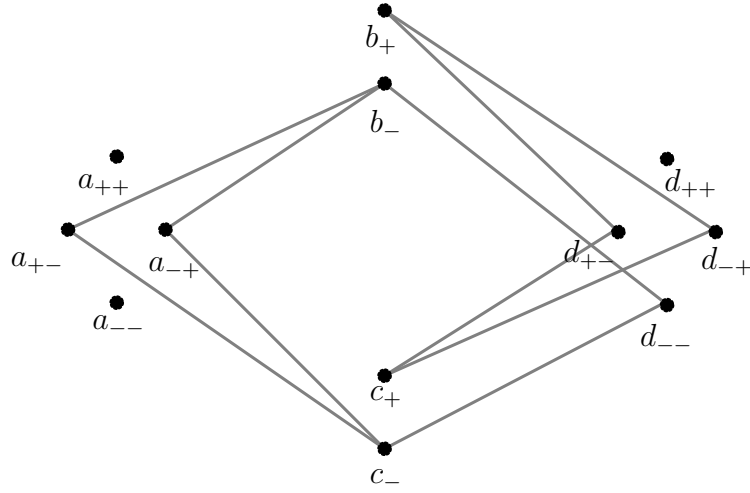


Figure 2.22: The cube of chains for the annular Hopf link from Figure 2.7, with AKh differentials drawn as solid lines and $\tau^\#$ drawn as dotted lines. The arrow heads have been dropped; gr_i increases to the right. This is a finer version of Figure 2.9.

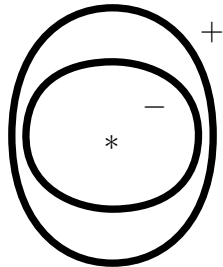


Figure 2.23: The marked resolution corresponding to the generator a_{+-} .

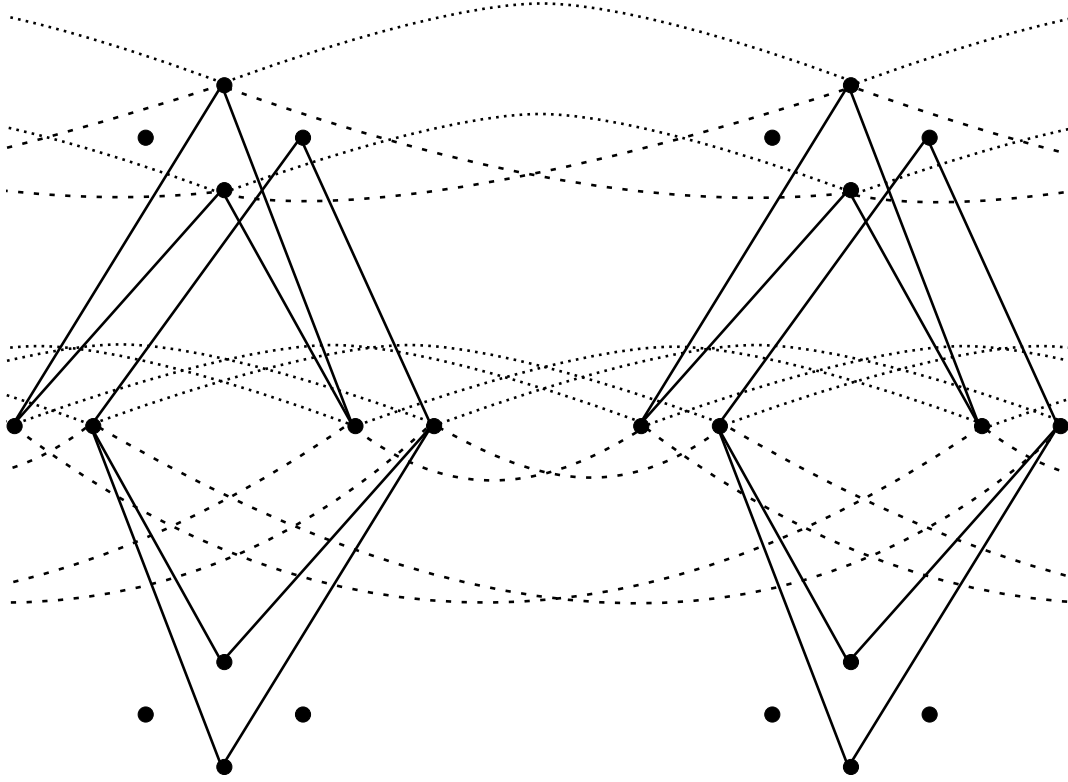


Figure 2.24: A local picture of the $\text{AKH}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$ bicomplex for the Hopf link example. There are two copies of the cube of chains in this figure, rotated 90° clockwise from from Figure 2.9. The dotted lines which bend up represent the “identity” map which increases the t -grading, and the dashed lines which bend down represent the $\tau^\#$ map. This local picture is repeated horizontally to form $\text{AKH}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$.

Putting this all together, the bicomplex $\text{AKH}^{\text{Tate}}(\mathcal{D}(\tilde{L}))$ locally looks like Figure 2.24.

After cancelling all the $\tau^\#$ arrows, the only remaining interesting page to compute the homology of is ${}^{hv}E^2$, shown in Figure 2.25. As expected, each slope-(-2) line in ${}^{hv}E^2$ corresponds to the cube of chains for the quotient link L , under the correspondence described by Propositions 2.29 and 2.30.

In this case, the fact that there are no higher differentials is clear from the fact that $CKh(\mathcal{D}(\tilde{L}))$ is only supported on three adjacent i -gradings. For comparison, the annular Khovanov homologies for \tilde{L} and L are listed in the following chart.

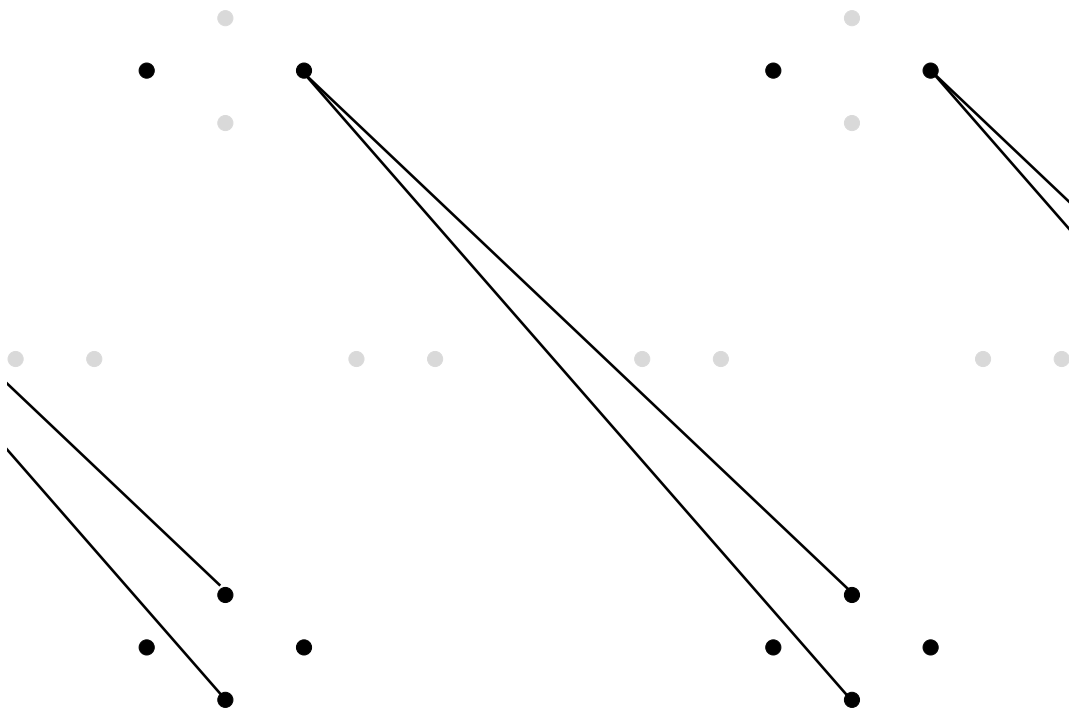


Figure 2.25: A local picture of the page $^{hv}E^2$ in our Hopf link example.

(j, k)	$\text{rk}_{\mathbb{F}_2}(AKh^{j,k}(L))$	$(2j - k, k)$	$\text{rk}_{\mathbb{F}_2}(AKh_{2j-k,k}(\tilde{L}))$
$(3,2)$	1	$(4,2)$	1
$(1,0)$	1	$(2,0)$	1
$(-1, -2)$	1	$(0,-2)$	1
$(3,0)$	1	$(6,0)$	1
		$(4,0)$	2

2.7 Decategorification

Annular Khovanov homology was originally described as a categorification of the Kauffman bracket skein module in the trivial I -bundle over the annulus. In our case, we work with \mathbb{F}_2 coefficients and follow the conventions in [Rob13].

Consider the $\mathbb{F}_2[q^{\pm 1}]$ -module consisting of all links in $A \times I$ modulo the usual Kauffman bracket skein relations. The set of all trivial braid closures $\mathcal{B} = \{\widehat{\mathbf{1}}_s \mid s = 0, 1, 2, \dots\}$ form a canonical basis for the skein module.

We use the variables t, q , and x to record the i -, j -, and k -gradings, respectively. Setting $q = A^{-2}$ and introducing the k -grading to the skein relation, we have a variant of the Kauffman bracket skein relations. Choose a particular crossing c and let L_0 and L_1 denote the links obtained from L by smoothing c by the 0-resolution and 1-resolution, respectively. Then our relations are

$$\begin{aligned}\langle L \rangle &= \langle L_0 \rangle + tq\langle L_1 \rangle \\ \langle L \cup U \rangle &= (q + q^{-1})\langle L \rangle\end{aligned}$$

where U is an annular unknot (a trivial circle).

Normalizing by setting $\widehat{\mathbf{1}}_1 \mapsto qx + q^{-1}x^{-1}$, we obtain a map

$$\phi : \mathbb{F}_2[t, q^{\pm 1}, \mathcal{B}] \rightarrow \mathbb{F}_2[t, q^{\pm 1}, x^{\pm 1}]$$

by viewing \mathcal{B} as a monoid under disjoint union. Finally, we add in the proper grading shifts to obtain a variant of the Jones polynomial over \mathbb{F}_2 by setting $V(t, q, x) = t^{n_-} q^{n_+ - 2n_-} \phi(L)$. (To be clear, one needs to first orient L to obtain n_- and n_+ , and we are viewing the link L as an element of the skein module.)

Of interest to us is the annular link invariant $V_L(-1, q, x)$ which is the graded Euler characteristic of $AKh(L)$. (See Theorem 8.1 in [APS04] for details.) In other words,

$$V_L(t, q, x) = \sum_{i,j,k} t^i q^j x^k \text{rk}_{\mathbb{F}_2}(AKh^{i,j,k}(L))$$

so the graded Euler characteristic is

$$V_L(-1, q, x) = \sum_{i,j,k} (-1)^i q^j x^k \text{rk}_{\mathbb{F}_2}(AKh^{i,j,k}(L)).$$

As in [APS04] and [Rob13], we view the Kauffman bracket as a set of polynomials by defining $q_{k,L}(t, q)$ to be the coefficient of x^k in $V(t, q, x)$, so that

$$q_{k,L}(-1, q) = \sum_{i,j} (-1)^i q^j \text{rk}_{\mathbb{F}_2} AKh^{i,j,k}(L)$$

is the coefficient of x^k in the decategorification of $AKh(L)$.

The decategorification of Theorem 2.1 is the following:

Corollary 2.3. *For all j and k ,*

$$\langle q_{k,\tilde{L}}(-1, q), q^{2j-k} \rangle \equiv \langle q_{k,L}(-1, q), q^j \rangle \pmod{2}$$

where $\langle f, g \rangle$ denotes the coefficient of g in f .

Proof. This follows from Theorem 2.1 and the fact that cancellation reduces rank by a multiple of 2 at each step (represented by erasing the dots at the head and tail of the arrow). \square

In [Mur71], Murasugi proved a relationship between the Alexander polynomials of a periodic link and that of its quotient; in [Mur88], he proved an analogous formula for the Jones polynomial. By considering Theorem 2.1 with respect to the $j_1 = j - k$ grading rather than the j - and k -gradings individually, we obtain an annular analogue to Murasugi's formulas:

Corollary 2.4. $V_{\tilde{L}}(1, q, q^{-1}) \equiv [V_L(1, q, q^{-1})]^2 \pmod{2}$.

If on the other hand one is more interested in the usual Jones polynomial of the 2-periodic link, we may forget the k -grading information by setting $x = 1$, obtaining the following relationship:

Corollary 2.5. $V_{\tilde{L}}(1, q, 1) \equiv V_L(1, q^2, q^{-1}) \pmod{2}$.

It is then natural to ask if there is a categorification of this relationship, coming from a spectral sequence from the Khovanov homology of \tilde{L} to the annular Khovanov homology of L (with some grading information sacrificed). In the next section, we show that this spectral sequence indeed exists for some families of annular links, and is likely to exist in general by way of a bicomplex very similar to the AKH^{Tate} bicomplex.

2.8 The Khovanov-Tate bicomplex

We can similarly define the *Khovanov-Tate bicomplex* $\mathrm{KH}^{\mathrm{Tate}}(\mathcal{D}(\tilde{L}))$ by replacing the vertical differentials ∂_{AKh} in $\mathrm{AKH}^{\mathrm{Tate}}(\mathcal{D}(\tilde{L}))$ with the Khovanov differential ∂_{Kh} .

Conjecture 2.6. *Let \tilde{L} be a 2-periodic link in S^3 with quotient link L . There is a spectral sequence with*

$$E^1 \cong Kh(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}] \Rightarrow E^\infty \cong AKh(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}].$$

This would in turn imply the following cascade of rank inequalities:

$$rk_{\mathbb{F}_2} AKh(\tilde{L}) \geq rk_{\mathbb{F}_2} Kh(\tilde{L}) \geq rk_{\mathbb{F}_2} AKh(L) \geq rk_{\mathbb{F}_2} Kh(L)$$

where the first and third inequalities are given by the k -grading filtration on $CKh(\mathcal{D}(\tilde{L}))$ and $CKh(\mathcal{D}(L))$.

In §2.8.1 we give abbreviated proofs of the Kh analogues of the results from the AKh case which support Conjecture 2.6. But first, we discuss holistic reasons why we would expect this conjecture to hold.

Remark 2.41. Aside from the proofs of theorems given in §2.8.1, this conjecture would not be hard to believe. $H_*(Tot(\mathrm{KH}^{\mathrm{Tate}}))$ is supposed to capture information about both the periodic link and the involution witnessing the symmetry, so the presence of the braid axis in S^3 is inherently captured in the data of the $\mathrm{KH}^{\mathrm{Tate}}$ bicomplex. Hence $H_*(Tot(\mathrm{KH}^{\mathrm{Tate}}))$ *should* take on the form of an annular version of Khovanov homology. Furthermore, Seidel and Smith proved the rank inequality for symplectic Khovanov homology [SS10], which is known to be equivalent to Khovanov homology over ground

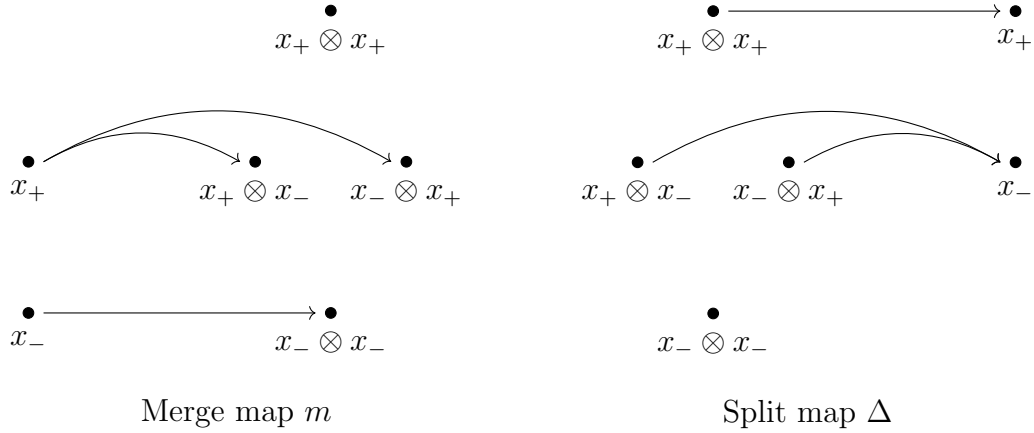


Figure 2.26: The Khovanov differentials. These do not depend on the basepoint $*$, so x can stand for v or w .

rings of characteristic 0 [ASmi]:

$$\mathrm{rk}_{\mathbb{F}_2} Kh_{\mathrm{symp}}(\tilde{L}) \geq \mathrm{rk}_{\mathbb{F}_2} Kh_{\mathrm{symp}}(L).$$

They conjecture that symplectic Khovanov homology is also isomorphic to Khovanov homology in characteristic 2 [SS06], so some of the rank inequalities implied by this conjecture agree with their result.

2.8.1 Analogous statements in the KH^{Tate} case

Khovanov homology can be viewed as a deformation of annular Khovanov homology, defined for knots in S^3 . This is why we have been denoting the underlying vector space for the annular Khovanov complex of a link diagram by CKh ; this is the same underlying vector space as in the Khovanov complex. Since there is no basepoint in the link diagram, the Khovanov distinguished generators are only doubly graded, by the homological i -grading and the quantum j -grading. There are only two types of edge maps: merge and split. Their definitions are shown in Figure 2.26 using the dot and arrow notation we used to describe the AKh differentials in back in §2.3.3.

Let ${}^{hv}E$ and ${}^{vh}E$ denote the spectral sequences induced by the row-wise and column-wise filtrations, respectively, of the bicomplex $\mathrm{KH}^{\mathrm{Tate}}(\mathcal{D}(\tilde{L})) = CKh(\mathcal{D}(\tilde{L})) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$, where $\partial^v = \partial_{Kh}$ and $\partial^h = 1 + \tau^\#$.

Lemma 2.42. ∂_{Kh} commutes with $\tau^\#$.

Proof. We need to check that the two types of edge maps (merge and split) commute with $\tau^\#$. This is easily checked, and clear from thinking about the involution acting on the saddle cobordism representing the merge or split map, just as in the AKh cases. □

Lemma 2.43. ${}^{vh}E^1 \cong Kh(\tilde{L}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$.

Proof. This follows from the same argument as in the proof of Lemma 2.23. □

Lemma 2.44. ${}^{hv}E^1$ is generated by the equivariant generators of $\mathrm{KH}^{\mathrm{Tate}}(\mathcal{D}(\tilde{L}))$. Hence if r is odd, ${}^{hv}d^r = 0$.

Proof. Since the rows of the $\mathrm{KH}^{\mathrm{Tate}}(\mathcal{D}(\tilde{L}))$ bicomplex are identical to those of the $\mathrm{AKH}^{\mathrm{Tate}}(\mathcal{D}(\tilde{L}))$ complex, this is equivalent to Lemma 2.24. The second statement follows from the first just as in the proof of Lemma 2.26. □

Lemma 2.45. Every component of ${}^{hv}d^2$ is induced by the cancellation of a $\tau^\#$ arrow at a row corresponding to an odd Hamming weight.

Proof. The proof of Lemma 2.27 also holds in this case. □

Proposition 2.46. There is a one-to-one correspondence

$$\begin{aligned} \{\text{generators of } {}^{hv}E^2\} &\longleftrightarrow \{\text{generators of } CKh(L) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]\} \\ \tilde{v} &\longleftrightarrow v \end{aligned}$$

induced by τ .

Proof. By Lemma 2.44, we have already identified the generators of ${}^{hv}E^2$ with the equivariant marked resolutions, which the proof of Proposition 2.29 shows to be in bijection with the generators of $CKh(\mathcal{D}(L)) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\theta, \theta^{-1}]$ (since this is the underlying vector space in both the $KH^{\text{Tate}}(\mathcal{D}(L))$ and $AKH^{\text{Tate}}(\mathcal{D}(L))$ bicomplexes). \square

Proposition 2.47. *Under the correspondence above, each slope -2 line in the bicomplex $({}^{hv}E^2, {}^{hv}d^2)$ is isomorphic to the complex $(CKh(\mathcal{D}(L)), \partial_{Kh})$.*

Proof. Again, we just need to compute the induced length 2 differentials. Since there are six types of AKh differentials, we can verify them case-by-case. The computations are shown in Figures 2.27 through 2.32. \square

Since the proof that higher differentials vanish relies on the k -grading, we cannot use the same methods to determine whether higher differentials vanish in the KH^{Tate} case. However, the k -grading is only needed when considering atoms of type \mathcal{E} (see §2.5.4), so one can still consider special cases where this obstruction is not needed for the conjecture to hold.

2.8.2 Positive and negative braid closures

Recall that the k -grading is used in the proof of Theorem 2.1 only when the j -grading was insufficient, namely in the situation where all the first equivariant steps through the subcube are \mathcal{E} atoms. From this we deduce that Conjecture 2.6 holds for “mostly negative” links.

Theorem 2.7. *If the annular braid closure $L = \widehat{\beta}$ has a diagram with at most 1 positive crossing, then the spectral sequence in Conjecture 2.6 exists and the cascade of rank inequalities holds.*

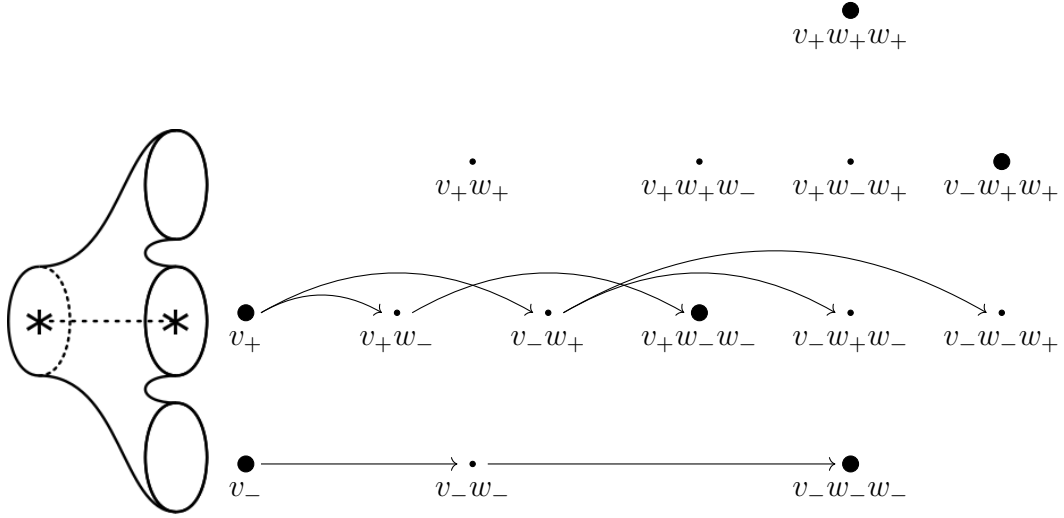


Figure 2.27: Type \mathcal{A} : $\tilde{v} \rightarrow \tilde{v}\tilde{w}$ for Khovanov differentials.

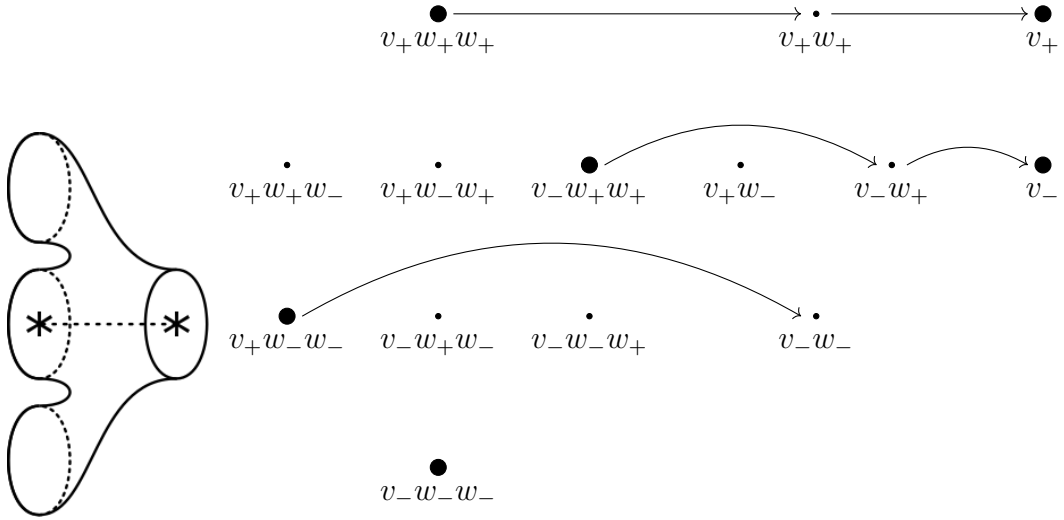


Figure 2.28: Type \mathcal{D} : $\tilde{v}\tilde{w} \rightarrow \tilde{v}$ for Khovanov differentials.

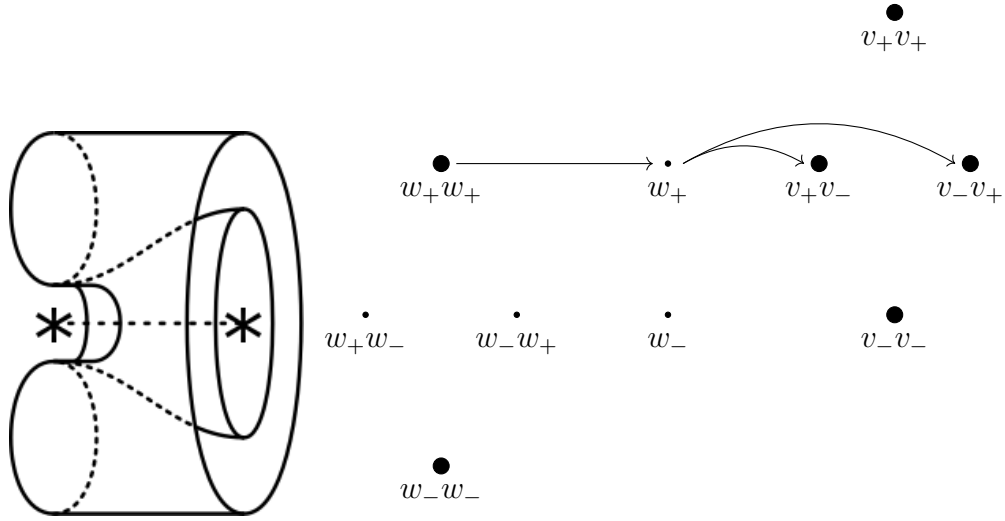


Figure 2.29: Type \mathcal{B} : $\tilde{w} \rightarrow \tilde{v}\tilde{v}$ for Khovanov differentials.

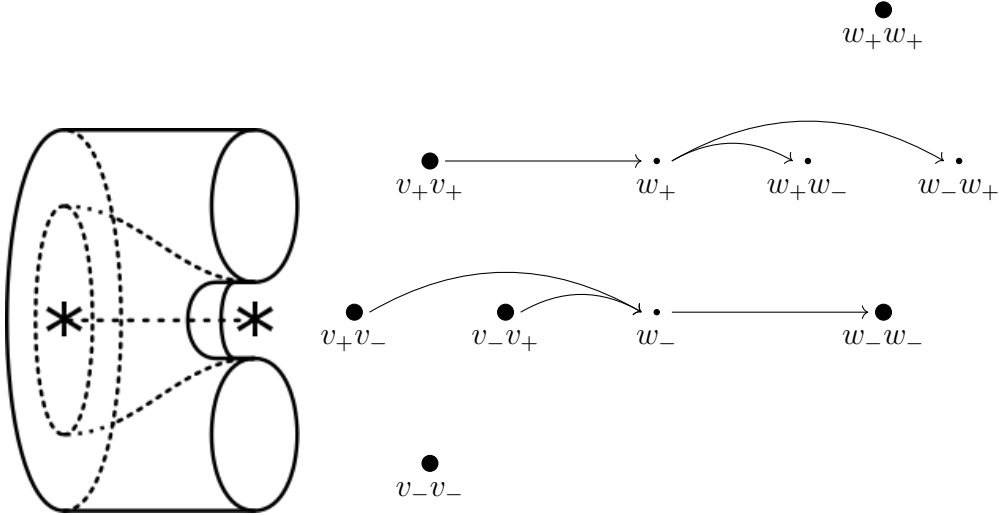


Figure 2.30: Type \mathcal{E} : $\tilde{v}\tilde{v} \rightarrow \tilde{w}$ for Khovanov differentials.

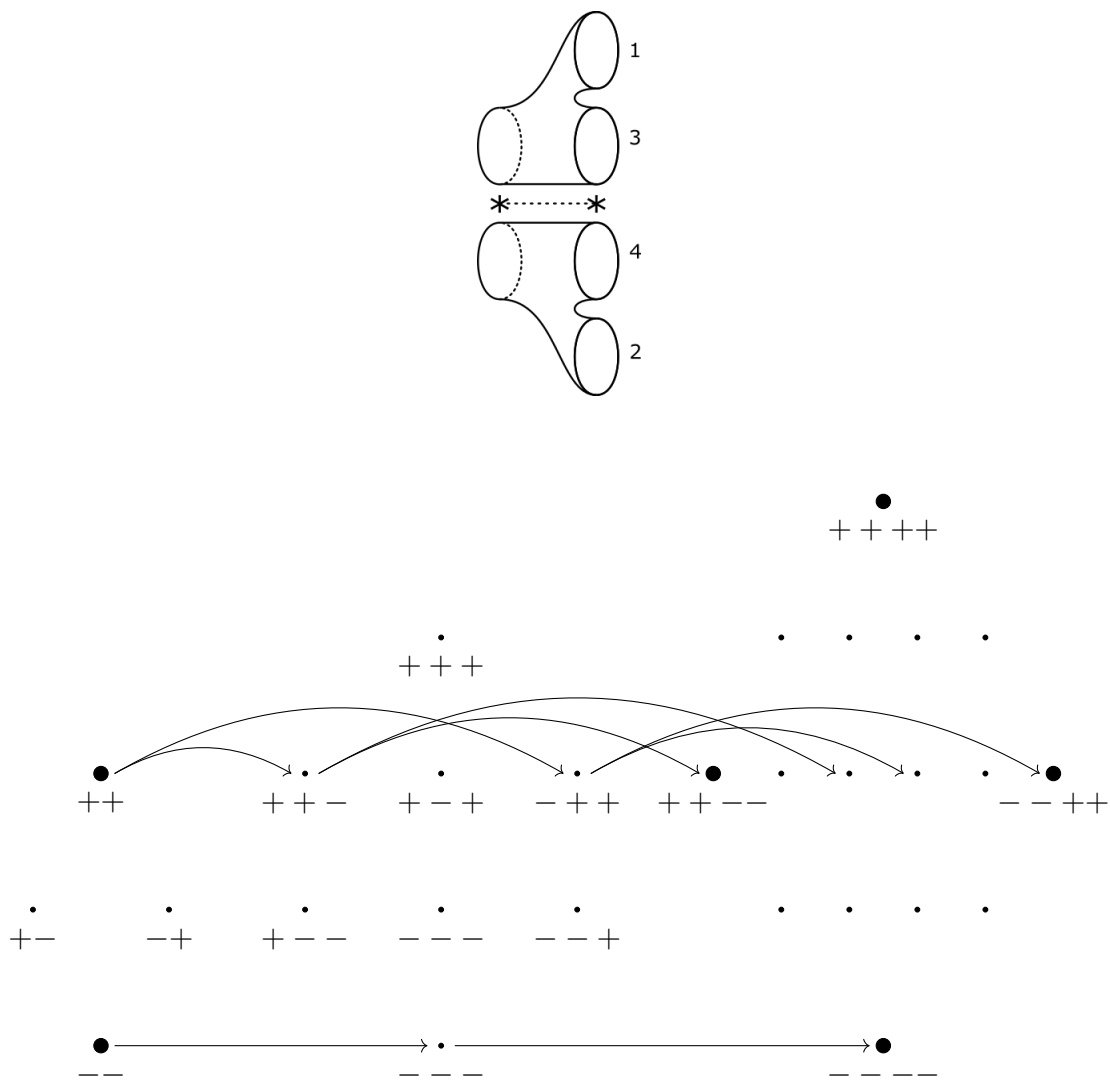


Figure 2.31: Type \mathcal{C} : $\tilde{w} \rightarrow \tilde{w}\tilde{w}$ for Khovanov differentials.

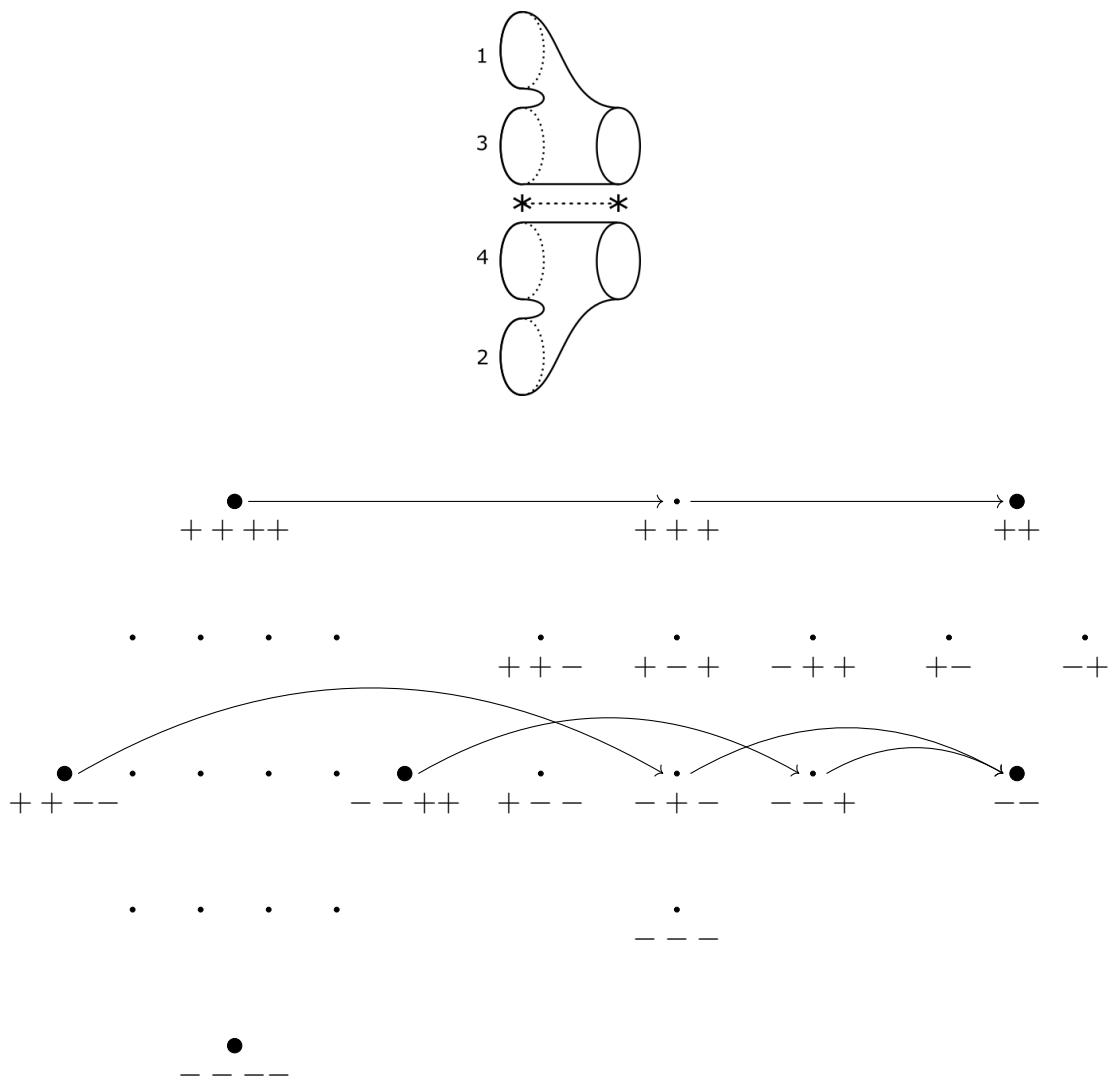


Figure 2.32: Type F: $\tilde{w}\tilde{w} \rightarrow \tilde{w}$ for Khovanov differentials.

Proof. The proof of Theorem 2.1 relies mostly on the j -grading to prove that the ${}^{hv}E$ spectral sequence collapses on page ${}^{hv}E^3$, except for the case of the generator v_+v_+ in the starting resolution of Type \mathcal{E} atoms. In the KH^{Tate} case, again the j -grading is sufficient for all cases except for Type \mathcal{E} , so for cases where at most one atom of Type \mathcal{E} appears, longer differentials cannot exist. This corresponds to the case where $\text{Cube}(\mathcal{D}(L))$ has at most one edge corresponding to a change of resolution from two nontrivial circles to two trivial circles. This means that the all-zeros resolution of $\mathcal{D}(L)$ corresponds to at most one braid-like resolution; that is, the braid β has at most one positive crossing. \square

On the other hand, we cannot use the same method to prove the existence of the spectral sequence for “mostly positive” braid closures. However, we can use duality in Khovanov and annular Khovanov homology to show that the cascade of rank inequalities still holds in this case.

Corollary 2.8. *If the annular braid closure $L = \widehat{\beta}$ has a diagram with at most 1 negative crossing, then the cascade of rank inequalities holds.*

Proof. Let L be a link in S^3 and \mathcal{D} a diagram of L . Let $L^!$ denote the mirror of L , and $\mathcal{D}^!$ the dual diagram to \mathcal{D} (switch the sign of all crossings). Observe that the complex $(CKh(\mathcal{D}), \partial^{Kh})$ is dual to $(CKh(\mathcal{D}^!), \partial^{Kh})$. The cohomology of $CKh(\mathcal{D})$ is isomorphic to the homology of $CKh(\mathcal{D}^!)$, which in turn is isomorphic to the cohomology of $CKh(\mathcal{D}^!)$, as we are working over field coefficients. (It is now important to note that what we have been calling “Khovanov homology” is actually a cohomology theory.) Therefore $rk_{\mathbb{F}_2}Kh(\mathcal{D}) = rk_{\mathbb{F}_2}Kh(\mathcal{D}^!)$. For an annular link $L \subset A \times I$, the above paragraph holds analogously for annular Khovanov (co)homology. Therefore it follows from Theorem 2.7 that the cascade of rank inequalities also holds for mostly positive braid closures. \square

Aside from proving the conjecture, many related questions remain.

Question 2.48. Are there other families of links for which the conjecture holds? Do certain steps in the proof for the $\mathrm{AKH}^{\mathrm{Tate}}$ spectral sequence point to other obstructions to longer differentials for the $\mathrm{KH}^{\mathrm{Tate}}$ spectral sequence? What are some other algebraic conditions on a braid closure which guarantee that the conjecture holds?

Question 2.49. While Khovanov-thin links have been studied for some time, at present I am unaware of any explorations of “annular Khovanov-thin links.” One could define an annular link L to be *AKh-thin* if for each pair of quantum and \mathfrak{sl}_2 weight space gradings (j, k) , $\mathrm{AKh}^{j,k}$ is supported on two adjacent i -gradings. Then *AKh-thinness* of the quotient link L suffices for the spectral sequence in Conjecture 2.6, since all differentials after page ${}^{hv}E^3$ are too long in i -degree to be nontrivial. Which classes of knots or links are *AKh-thin*?

2.8.3 Obstructing 2-periodicity

One motivation for pursuing Conjecture 2.6 arises from the study of obstructions to periodicity in links. The present method may overcome the following difficulties encountered by prior work.

First, obstructing p -periodicity for small p is particularly difficult. In particular, Murasugi’s Jones and Alexander polynomial congruences [Mur71, Mur88] are weaker for $p = 2$ as mod 2 congruences only detect parity differences in coefficients. Comparing ranks in categorified invariants can potentially detect more differences. Borodzik and Poltarczyk choose to work with the Khovanov polynomial rather than a graded Euler characteristic for this reason (see page 2 of [BP17]); however, their obstruction is only valid for $p > 3$.

Second, sometimes obstructions to knot periodicity do not extend to links with multiple components. For example, Hendricks’ rank inequality for \widehat{HFK} pertains only to knots (see §3.2 in [Hen15]). The present methods do not depend on the number of link

components.

In this section, we formulate some frameworks for obstructing 2-periodicity as consequences of Theorem 2.1 and of various portions of Conjecture 2.6, including the special cases of Theorem 2.7 and Corollary 2.8.

Notation 1. *Throughout this section, we fix the following notation. The symbols L and \tilde{L} indicate annular links (in $A \times I$), with $[L]$ and $[\tilde{L}]$ denoting their isotopy classes as links in S^3 . Since our frameworks consider a potentially periodic link $\tilde{L} = \widehat{T}$ or $[\tilde{L}]$ and compare it with a potential quotient link L or $[L]$, a priori we do not know whether $\tilde{L} = \widehat{T^2}$.*

Corollary 2.2 can be framed as an obstruction as follows.

Corollary 2.50. *Let $L = \widehat{T}$ and \tilde{L} be annular links. Suppose there is some (j, k) bigrading for which $rk_{\mathbb{F}_2} AKh^{2j-k, k}(\tilde{L}) \leq rk_{\mathbb{F}_2} AKh^{j, k}(L)$. Then $\tilde{L} \neq \widehat{T^2}$.*

Now suppose a weaker version of Conjecture 2.6 were true; that is, suppose that the rank inequality holds for Khovanov homology. (Recall that Seidel and Smith proved this rank inequality holds for symplectic Khovanov homology over \mathbb{F}_2 [SS10], which is to date still only conjectured to be equivalent to Khovanov homology over \mathbb{F}_2 .)

Conjecture 2.51. *Let $[L]$ and $[\tilde{L}]$ be two links in S^3 . If $rk_{\mathbb{F}_2} Kh([\tilde{L}]) \leq rk_{\mathbb{F}_2} Kh([L])$, then $[\tilde{L}]$ is not 2-periodic with quotient link $[L]$. In other words, there are no representatives $\tilde{L} \in [\tilde{L}]$ and $L \in [L]$ such that $L = \widehat{T}$ and $\tilde{L} = \widehat{T^2}$.*

Remark 2.52. By comparing Propositions 2.29 and 2.46, it is clear that the present method will not yield a rank inequality for Khovanov homology splitting along any gradings, because the correspondence between equivariant generators of $CKh(\mathcal{D}(\tilde{L}))$ and the generators of $CKh(\mathcal{D}(L))$ indicates a dependence on the position of the basepoint via the k -grading. So, whenever we use rank inequalities involving Khovanov homology, we must use the rank of the entire homology.

For example, we deduce the following obstruction from Theorem 2.7 (resp. Corollary 2.8):

Corollary 2.53. *Let $L = \widehat{\beta}$ be an annular link which admits a braided annular diagram $\mathcal{D}(\widehat{\beta})$ with at most one negative (resp. positive) crossing.*

- *If $rk_{\mathbb{F}_2}Kh([\tilde{L}]) \leq rk_{\mathbb{F}_2}Kh([L])$, then $[\tilde{L}]$ is not 2-periodic with quotient link $[L]$.*
- *If $rk_{\mathbb{F}_2}Kh([\tilde{L}]) \leq rk_{\mathbb{F}_2}AKh(L)$, then $[\tilde{L}] \neq [\widehat{\beta^2}]$.*

Chapter 3

Localization in Khovanov homology

3.1 Introduction

3.1.1 Motivation and results

In [Kho00] Khovanov categorified the Jones polynomial: to a link diagram L , he associated a bigraded chain complex, whose graded Euler characteristic is (a certain normalization of) the Jones polynomial of L , and whose (graded) chain homotopy type is an invariant of the underlying link. Several generalizations were soon constructed; for example, [Kho02], [BN05] developed theories for tangles. Ozsváth-Rasmussen-Szabó [ORSz13] constructed a version, *odd Khovanov homology*, also categorifying the Jones polynomial, and agreeing with Khovanov homology over the field of two elements. A further generalization, *annular Khovanov homology*, an invariant of links in the thickened annulus, was introduced by Asaeda-Przytycki-Sikora [APS06], and this generalizes readily to give also *odd annular Khovanov homology*, as in [GW18]. Other generalizations, for other polynomials, were given by [KR08a, KR08b] and others, and have since been extensively developed.

The purpose of the present paper is to investigate the structure of Khovanov homology

in the presence of symmetry; that is, we study the Khovanov homology of *periodic links*. We say that a link $\tilde{L} \subset S^3$ is *p-periodic* if there is a $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ -action on (S^3, \tilde{L}) which preserves \tilde{L} and whose fixed-point set is an unknot \tilde{U} disjoint from \tilde{L} . A particular application of the techniques of this paper is the following:

Theorem 3.1. *Let \tilde{L} be a p^n -periodic link, for a prime p , with quotient link L . Let $Kh(\tilde{L}; \mathbb{F}_p)$ (resp. $Kh_o(\tilde{L}; \mathbb{F}_p)$) denote the Khovanov homology (resp. odd Khovanov homology) of \tilde{L} , with coefficients in \mathbb{F}_p , the field of p elements. Let $AKh(L; \mathbb{F}_p)$ (resp. $AKh_o(L; \mathbb{F}_p)$) denote the (resp. odd) annular Khovanov homology of L , viewed in the complement of $U = \tilde{U}/\mathbb{Z}_p$. Then,*

$$\dim Kh(\tilde{L}; \mathbb{F}_p) \geq \dim AKh(L; \mathbb{F}_p) \quad \text{and} \quad \dim Kh_o(\tilde{L}; \mathbb{F}_p) \geq \dim AKh_o(L; \mathbb{F}_p).$$

The motivation for this study comes from the both the application of classical Smith theory to Floer theories, and the general perspective of studying Floer and Khovanov invariants via the (often only conjectural) spectra underlying these theories.

Let G be a group of order p^n with p prime, acting on a finite-dimensional topological space M , with fixed-point set M^G . A version of the classical Smith inequality states [Smi38], [Bre67]:

$$\dim H^*(M; \mathbb{F}_p) \geq \dim H^*(M^G; \mathbb{F}_p). \quad (3.2)$$

In low-dimensional topology and symplectic geometry, many results have been developed in analogy with the Smith inequality, relating the Floer homology of some object with symmetries with the Floer homology of its ‘quotient,’ when the latter notion makes sense. In particular, Seidel-Smith [?] proved an analogue of the Smith inequality for $p = 2$ in Lagrangian Floer theory. In fact, one of the motivations for [?] was its application to symplectic Khovanov homology: Seidel-Smith prove a localization result for symplectic Khovanov homology [SS06] of 2-periodic links. Seidel-Smith further remark

in [?] that the combinatorial analogue to their symplectic Khovanov rank inequality was not known to hold at the time; Corollary 3.87 (a consequence of Theorem 3.1) of the present paper asserts that this analogue does indeed hold. (Note that Khovanov homology and symplectic Khovanov homology are known to agree in characteristic 0 by work of Abouzaid-Smith [ASmi], but Smith-type inequalities from \mathbb{Z}_p -localization only hold in finite characteristic.)

The Seidel-Smith inequality led to many further developments in low-dimensional topology. For instance, Hendricks [Hen12] showed that the knot Floer homology of a knot $K \subset S^3$ has rank at most as large as that of the knot Floer homology of the preimage \hat{K} in the branched double cover $\Sigma(K)$, and also obtained relationships between knot Floer homology of 2-periodic knots and that of their quotients [Hen15] (see also [HLS16] and [Boy18]).

From the perspective of the present paper, the Seidel-Smith inequality reflects the extent to which Floer theories contain more information than just the resulting chain complex (indeed, the Smith inequality is a fact about *spaces*, not about chain complexes). A particularly striking formulation of this principle is found in Lidman-Manolescu [LM18b], where they showed that, roughly, for a p^n -sheeted regular cover $\pi: \tilde{Y} \rightarrow Y$ there is an action of a group G of order p^n on the Seiberg-Witten Floer space $SWF(\tilde{Y}; \pi^*\mathfrak{s})$, so that the fixed-point set is $SWF(Y, \mathfrak{s})$, the Seiberg-Witten Floer space of the quotient. They thus obtain a rank inequality, by applying the classical Smith inequality:

$$\sum_i \dim \tilde{H}_i(SWF(\tilde{Y}, \pi^*\mathfrak{s}); \mathbb{F}_p) \geq \sum_i \dim \tilde{H}_i(SWF(Y, \mathfrak{s}); \mathbb{F}_p).$$

Recall that Lidman-Manolescu [LM18a] identified the reduced homology of $SWF(Y, \mathfrak{s})$ with the tilde flavor of monopole Floer homology $\widetilde{HM}_*(Y, \mathfrak{s})$. Further, Colin-Ghiggini-Honda and Kutluhan-Lee-Taubes [CGH11],[KLT10] proved $\widetilde{HM}_*(Y, \mathfrak{s}) = \widehat{HF}_*(Y, \mathfrak{s})$. Then the result of [LM18b] gives an inequality of ranks of Heegaard-Floer homology,

and in particular strong constraints on L -spaces arising as regular covers.

In the present paper, we relate Khovanov space-level invariants of a periodic link \tilde{L} with those of the quotient link L . This space-level relationship leads to a relationship on the level of homology that does not seem to follow in a simple way from the chain complex description of Khovanov homology. A priori, it is difficult to relate any given Khovanov chain complex of a periodic link with any given Khovanov chain complex of the quotient, since without further information these are just chain complexes without further structure. However, the second author showed in [Zha18], without using space-level invariants, that there is a spectral sequence relating the annular Khovanov homology of a periodic link with that of its quotient. This took advantage of a bonus grading in annular Khovanov homology, which is a richer invariant than Khovanov homology itself [GLW18]; the extra structure was essential to that result.

To set up notation, recall that for a link $L \subset S^3$, Lipshitz-Sarkar [LS14] constructed a CW spectrum $\mathcal{X}_e(L)$ whose stable homotopy type is an invariant of the underlying link L , and whose reduced cellular chain complex is precisely the Khovanov chain complex $Kc(L)$. Their construction readily generalizes to produce an annular Khovanov spectrum of a link L in the thickened annulus (see also [LOS]). Further, in [SSS18], a family $\mathcal{X}_n(L)$ of CW spectra was constructed for $n \in \mathbb{Z}_{\geq 0}$, so that $\mathcal{X}_0(L) = \mathcal{X}_e(L)$ and so that the reduced cellular chain complex $\tilde{C}_{\text{cell}}(\mathcal{X}_n(L))$ is the even Khovanov chain complex $Kc(L)$ for n even, and is the odd Khovanov chain complex $Kc_o(L)$ for n odd. It is again straightforward to construct an annular Khovanov spectrum $\mathcal{AKH}_n(L)$ for any $n \in \mathbb{Z}_{\geq 0}$, whose reduced cellular chain complex $\tilde{C}_{\text{cell}}(\mathcal{AKH}_n(L))$ is the even annular Khovanov chain complex $AKc(L)$ if n is even, and the odd annular Khovanov chain complex $AKc_o(L)$ if n is odd. The Khovanov spaces and spectra split as a wedge sum according to quantum, and in the case of annular, (k) -gradings, as $\mathcal{X}_n(L) = \vee_j \mathcal{X}_n^j(L)$ and $\mathcal{AKH}_n(L) = \vee_{j,k} \mathcal{AKH}_n^{j,k}(L)$. Furthermore, for $n \geq 1$, $\mathcal{X}_n(L)$ (resp. $\mathcal{AKH}_n(L)$) is a \mathbb{Z}_2 -

equivariant spectrum with geometric fixed points $\Sigma^{-1}\mathcal{X}_{n-1}(L)$ (resp. $\Sigma^{-1}\mathcal{AKH}_{n-1}(L)$).

The main result of the present paper is the following:

Theorem 3.3. *Fix $p > 1$ and let \tilde{L} be a p -periodic link with quotient link L . For each quantum grading j , $\mathcal{X}_0^j(\tilde{L})$ ($\mathcal{AKH}_0^{j,k}(\tilde{L})$) is naturally a \mathbb{Z}_p -equivariant spectrum, whose \mathbb{Z}_p -equivariant stable homotopy type is an invariant of the p -periodic link \tilde{L} (that is, the equivariant stable homotopy type is preserved by equivariant isotopies and equivariant Reidemeister moves of a diagram \tilde{D} of \tilde{L}). Further, the geometric fixed points are given by:*

$$\begin{aligned}\mathcal{X}_0^j(\tilde{L})^{\mathbb{Z}_p} &= \bigvee_{\{a,b|pa-(p-1)b=j\}} \mathcal{AKH}_0^{a,b}(L), \\ \mathcal{AKH}_0^{pj-(p-1)k,k}(\tilde{L})^{\mathbb{Z}_p} &= \mathcal{AKH}_0^{j,k}(L),\end{aligned}\tag{3.4}$$

up to suitable suspensions. Moreover, if $n \geq 1$ and p is odd, $\mathcal{X}_n^j(\tilde{L})$ ($\mathcal{AKH}_n^{j,k}(\tilde{L})$) is naturally a $\mathbb{Z}_2 \times \mathbb{Z}_p$ -equivariant spectrum, whose $\mathbb{Z}_2 \times \mathbb{Z}_p$ -equivariant stable homotopy type is an invariant of the p -periodic link \tilde{L} . Then, as \mathbb{Z}_2 -equivariant spectra,

$$\begin{aligned}\mathcal{X}_n^j(\tilde{L})^{\mathbb{Z}_p} &= \bigvee_{\{a,b|pa-(p-1)b=j\}} \mathcal{AKH}_n^{a,b}(L), \\ \mathcal{AKH}_n^{pj-(p-1)k,k}(\tilde{L})^{\mathbb{Z}_p} &= \mathcal{AKH}_n^{j,k}(L).\end{aligned}\tag{3.5}$$

Proof of Theorem 3.1: We begin by noting that, in Theorem 3.3, all the involved objects are suspension spectra of finite CW complexes, and the statements in Theorem 3.3 continue to hold at the level of the underlying CW complexes. Then, $\mathcal{X}_e(\tilde{L})$ (here, a finite CW complex) admits a \mathbb{Z}_{p^n} -action with fixed-point set $\mathcal{AKH}_0(L)$. The homology satisfies $\tilde{H}(\mathcal{X}_e(\tilde{L})) = Kh(\tilde{L})$, while $\tilde{H}(\mathcal{AKH}_0(L)) = AKh(L)$. Applying (3.2) to $M = \mathcal{X}_e(\tilde{L})$, Theorem 3.1 follows for the even case. The odd case is similar.

Further, we expect that the Tate spectral sequence arising from Theorem 3.3 should

be compatible with spectral sequences from Khovanov to Floer theories, perhaps being related to Hendricks' [Hen15], Roberts' [Rob13], or Xie's [Xie18] spectral sequences.

We mention a few further possible connections of Theorem 3.3 to other work. First, recall from [BPW19] that annular Khovanov homology of a link L can be realized as the Hochschild homology of an appropriate bimodule over the platform algebra (see [CK14],[Str05]). Recall moreover that Lawson-Lipshitz-Sarkar [LLS17b] have given a spectrum-level version of Khovanov's invariant for tangles [Kho02]. From these developments, it seems natural to conjecture that the annular Khovanov spectrum of a link is realized as the topological Hochschild homology of an appropriate spectral bimodule. If this conjecture holds, it is natural to ask whether and how the actions constructed in this paper pass over to give actions on the topological Hochschild homology. See also [LT16].

Note also that in a recent preprint, Borodzik-Politarczyk-Silvero [BPS18] use equivariant flow categories to also show that $\mathcal{X}_0(\tilde{L}) = \mathcal{X}_e(\tilde{L})$ admits a \mathbb{Z}_p -action; the main Theorem 1.2 of [BPS18] is the first sentence of Theorem 3.3 in the present paper, although it is not clear that the action constructed in [BPS18] and that constructed here (in the case $n = 0$) agree. In [BPS18], they further relate the Borel equivariant cohomology of $\mathcal{X}_e(\tilde{L})$ to Politarczyk's equivariant Khovanov homology [Pol15]. Jeff Musyt has also constructed a \mathbb{Z}_p -equivariant Khovanov stable homotopy type using methods similar to ours.

3.1.2 Techniques

We summarize the machinery and organization of this paper. This paper uses the machinery of Burnside functors, introduced in [HKK16] and [LLS], to study the Khovanov spectrum. The machinery of Burnside functors first appeared in [LLS] to handle the product formula for Khovanov spectra, by giving a construction of the Khovanov spec-

trum as a certain homotopy colimit, more convenient for many applications. We will use a slight generalization of Burnside functors from [SSS18], ‘decorated’ Burnside functors, introduced to generalize the construction of [LLS] to produce an odd Khovanov space. We first review the construction of [LLS], in order to explain what is done in the present paper.

In [LLS], the dual of the Khovanov chain complex of a link diagram with n ordered crossings is viewed as a diagram of abelian groups:

$$\mathfrak{F}_e: (\underline{2}^n)^{\text{op}} \rightarrow \mathbb{Z}\text{-Mod},$$

and similarly in [SSS18], the odd Khovanov chain complex is viewed as a diagram:

$$\mathfrak{F}_o: (\underline{2}^n)^{\text{op}} \rightarrow \mathbb{Z}\text{-Mod}$$

Let us recall, for K a finite group, the K -decorated *Burnside category* \mathcal{B}_K (written \mathcal{B} , if $K = \{1\}$), whose objects are finite sets, whose 1-morphisms are finite correspondences decorated by elements of K , and whose 2-morphisms are bijections respecting decorations. The 2-category \mathcal{B} naturally comes with a forgetful functor to abelian groups $\mathcal{B} \rightarrow \mathbb{Z}\text{-Mod}$ by sending a set S to the free abelian group $\mathbb{Z}\langle S \rangle$ generated by S . The Khovanov stable homotopy type arises from a lift, according to [LLS]:

$$\begin{array}{ccc} & & \mathcal{B} \\ & \nearrow \kappa\mathcal{H} & \downarrow \\ \underline{2}^n & \xrightarrow{\mathfrak{F}_e^{\text{op}}} & \mathbb{Z}\text{-Mod} \end{array}$$

On the other hand, given a homomorphism $\epsilon: K \rightarrow \mathbb{Z}_2$, there is a forgetful functor $\mathcal{B}_K \rightarrow \mathbb{Z}\text{-Mod}$, again by sending a set S to the free abelian group $\mathbb{Z}\langle S \rangle$ generated by S , and with $\mathbb{Z}\text{-Mod}$ -morphisms twisted by ϵ . The odd Khovanov stable homotopy type

arises from a lift:

$$\begin{array}{ccccc}
& & \mathcal{B}_{\mathbb{Z}_2} & & \\
& \nearrow \mathcal{KH}\mathcal{O} & \downarrow \epsilon = 0 & \searrow \epsilon = \text{Id} & \\
\mathbb{Z}^n & & \mathbb{Z}\text{-Mod} & & \mathbb{Z}\text{-Mod} \\
& \searrow \mathfrak{F}_e^{\text{op}} & & \nearrow \mathfrak{F}_o^{\text{op}} &
\end{array}$$

The even Burnside functor \mathcal{KH} is obtained by forgetting the \mathbb{Z}_2 -decorations on $\mathcal{KH}\mathcal{O}$.

Given a Burnside functor F , [LLS] gives a recipe, called *realization* (see Section 3.4), for how to construct a space, $||F_e(L)||$ (respectively, odd Khovanov space $||F_o(L)||$), as a homotopy colimit of a certain homotopy coherent diagram constructed from F . This is generalized in [SSS18], for the case of K nontrivial.

The goal of the present paper is to investigate extra structure on the realizations $||F||$ for, $F = \mathcal{KH}(\tilde{L})$ or $\mathcal{KH}\mathcal{O}(\tilde{L})$ for \tilde{L} p -periodic. A natural expectation is that $||F||$ should admit a \mathbb{Z}_p -action. The first technical work of the present paper consists of developing the correct notion of ‘actions’ on Burnside functors $F: \mathcal{C} \rightarrow \mathcal{B}_K$, for \mathcal{C} a small category, and on homotopy coherent diagrams $\mathcal{C} \rightarrow \text{Top}_*$, where Top_* is the category of pointed topological spaces.

First, we briefly explain the notion of ‘action’ on Burnside functors. A first guess is that a Burnside functor F with action should be a diagram $BG \times \mathcal{C} \rightarrow \mathcal{B}_K$, where BG is the category with one object, and morphisms G ; in analogy with viewing a pointed G -space as a diagram $BG \rightarrow \text{Top}_*$. The main technical difficulty is that, for the Khovanov-Burnside functor, $G = \mathbb{Z}_p$ acts on the category \mathcal{C} itself. We then define a notion of *external action* of a group G on a Burnside functor F as a kind of twist of the above definition. Alternatively, as in Remark 3.17, a Burnside functor with external action can be viewed as a functor from a thickening $\tilde{\mathcal{C}}$ of the category \mathcal{C} . In Section 3.3 we develop this notion.

We must next see how the realization process of [LLS] behaves on a Burnside functor F with action. As before, the problem is that we obtain a homotopy coherent diagram

where the index category itself admits a G -action (we call such a diagram a *diagram with external action* by G). Note that a homotopy coherent diagram with a G -action (so that G acts trivially on the index category) is simply a homotopy coherent diagram in the category of G -spaces, which would be readily handled along the lines of [SSS18].

In Section 3.5, we develop some machinery for homotopy colimits for homotopy coherent diagrams with an external action. We do not pursue the greatest level of generality here; indeed, a more satisfactory treatment would be to essentially generalize the bulk of [Vog73] to this situation. The main results are Proposition 3.50 and Lemma 5.6, while the main application to realizations of Burnside functors is Proposition 3.66. In fact, including Proposition 3.66 increases substantially the preliminaries we need, but is not needed in order to show that the Khovanov spaces of p -periodic links admit a \mathbb{Z}_p -action. Instead Proposition 3.66 is only needed to show that the resulting \mathbb{Z}_p -action is well-defined. In Section 3.6, we show that \mathcal{KH} and \mathcal{KHO} have external actions under suitable circumstances, and find the fixed point functors. This involves a reasonably detailed study of the relationship of resolution configurations in a periodic link with those in its quotient. It is somewhat interesting that the case of odd Khovanov homology here is substantially more involved than the even case.

We conclude the introduction with a few remarks. First, in sections dealing with homotopy coherent diagrams, we work with diagrams in K -spaces for a group K , although for all of our applications K will always be \mathbb{Z}_2 or trivial. We include the more general case because it is no more complicated, and also on account of a conjecture of [SSS18].

To explain this conjecture, recall that there are an infinite family of Khovanov spaces $\mathcal{X}_n(L)$ of a link L for $n \in \mathbb{Z}_{\geq 0}$, where the n -th space has cellular chain complex equal to the even (resp. odd) Khovanov chain complex if n is even (resp. odd). The conjecture of

[SSS18] is that there should be stable homotopy equivalences

$$\mathcal{X}_n(L) \simeq \mathcal{X}_{n+2}(L). \quad (3.6)$$

An attractive method of proving this conjecture would be the construction of a further Burnside functor $\mathcal{KH}_{\mathbb{Z}}: (\mathbb{Z}^n)^{\text{op}} \rightarrow \mathcal{B}_{\mathbb{Z}}$ recovering $\mathcal{KHO}(L)$ by taking $\mathbb{Z} \rightarrow \mathbb{Z}_2$. If such a functor could be constructed, the techniques of the present paper would apply immediately to its realizations. Note that even if (3.6) holds, Theorem 3.3 is not entirely boring for $n \geq 2$. Indeed, the statement (3.6) requires a choice of homotopy equivalence, and we expect that the natural family of homotopies realizing this equivalence (constructed from the putative $\mathcal{KH}_{\mathbb{Z}}$) is not contractible. That is, there may be no preferred homotopy equivalence $\mathcal{X}_n \rightarrow \mathcal{X}_{n+2}$.

We remark that we expect that much of this paper should generalize to give an action of the knot symmetry group on the (odd) Khovanov space.

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3.2 Khovanov homologies and periodic links

In this section, we briefly review the definition and basic properties of several Khovanov homology theories. For an oriented link $L \subset S^3$, we review the *even* Khovanov homol-

ogy $Kh(L) = Kh_e(L)$, defined by Khovanov [Kho00], and the *odd* Khovanov homology $Kh_o(L)$ defined by Ozsváth, Rasmussen and Szabó [ORSz13]. For an oriented link L in the thickened annulus $(S^1 \times [0, 1]) \times [0, 1]$, we review the annular Khovanov homology $Akh(L)$ defined by [APS06], as well as the odd annular Khovanov homology $AKh_o(L)$, which appeared in [GW18]. For a more detailed introduction to Khovanov homology, see [Kho00]. Our exposition follows [LLS] closely.

3.2.1 The cube category

We first recall the cube category. Call $\underline{2} = \{0, 1\}$ the one-dimensional cube, viewed as a partially ordered set by setting $1 > 0$, or as a category with a single non-identity morphism from 1 to 0.

Call $\underline{2}^n = \{0, 1\}^n$ the n -dimensional cube, with the partial order given by

$$u = (u_1, \dots, u_n) \geq v = (v_1, \dots, v_n) \text{ if and only if } \forall i (u_i \geq v_i).$$

It has the categorical structure induced by the partial order, where $\text{Hom}_{\underline{2}^n}(u, v)$ has a single element if $u \geq v$ and is empty otherwise. Write $\phi_{u,v}$ for the unique morphism $u \rightarrow v$ if it exists. The cube carries a grading given by $|v| = \sum_i v_i$. Write $u \geq_k v$ if $u \geq v$ and $|u| - |v| = k$. When $u \geq_1 v$, we call the corresponding morphism $\phi_{u,v}$ an *edge*, and call v an *immediate successor* of u .

We will study chain complexes refining the cube category whose homological gradings correspond to the gradings of the vertices. When we work with homotopy colimits, it is most useful for us to work with commutative cubes, i.e. cubes where the 2-dimensional faces commute. However, in order for $\partial^2 = 0$ to hold in the chain complex, we must assign signs to the edges of the cube to force each face to instead anticommute, leading to the following definition.

Definition 3.7. The *standard sign assignment* s is the following function from edges of $\underline{2}^n$ to \mathbb{F}_2 . For $u \geq_1 v$, let k be the unique element in $\{1, \dots, n\}$ with $u_k > v_k$. Then

$$s_{u,v} := \sum_{i=1}^{k-1} u_i \bmod 2.$$

Note that s may be viewed as a 1-cochain in $C_{\text{cell}}^*([0, 1]^n; \mathbb{F}_2)$. In general, $s + c$ is called a *sign assignment* for any 1-cocycle c in $C_{\text{cell}}^*([0, 1]^n; \mathbb{F}_2)$.

3.2.2 Even Khovanov homology Kh

Khovanov homology, introduced in [Kho00], is a combinatorial link invariant computed from a planar diagram of an oriented link by considering the *cube of resolutions*. The result is a bigraded homology theory associated to an oriented link. We sometimes refer to this theory as *even* Khovanov homology to distinguish it from odd Khovanov homology.

Let D be a link diagram with n ordered crossings. Each crossing \times can be *resolved* as the 0-resolution $\rangle \langle$ or the 1-resolution $\succ \prec$.

We will view Khovanov homology as coming from a functor

$$\mathfrak{F}_e : (\underline{2}^n)^{\text{op}} \longrightarrow \mathbb{Z}\text{-Mod}$$

which we define below. The theory is also defined similarly over more general rings. In the context of Smith inequalities (Subsection 3.6.5), we will use field coefficients.

Generators

For each $v \in \underline{2}^n$, let D_v be the complete resolution of D formed by taking the 0-resolution at the i^{th} crossing if $v_i = 0$, or the 1-resolution otherwise. The diagram D_v is a planar diagram of embedded circles. We write $Z(D_v)$ for the set of embedded circles (which

we just call circles) in D_v . A *Kauffman state* at v will be an element of the powerset of $Z(D_v)$. Let $\mathfrak{F}_e(v)$ be the free \mathbb{Z} -module generated by Kauffman states at v . We can think of Kauffman states as the monomials in the symmetric algebra generated by the circles $Z(D_v)$, modulo $x_i^2 = 0$ for each circle $x_i \in Z(D_v)$, that is, as an element of $\text{Sym}(Z(D_v))/(x^2)_{x \in Z(D_v)}$. Sometimes we will also think of a Kauffman state as a labeling $Z(D_v) \rightarrow \{x_-, x_+\}$ (for formal variables x_{\pm}), where the monomial $x_{i_1} \dots x_{i_k}$ corresponds to the labeling where the circle x_{i_j} is sent to x_- for all j , and the other circles are all sent to x_+ .

Arrows

Let $v, u \in \text{Ob}(\underline{2}^n)$ where $v \leq_1 u$. Since D_u and D_v differ only at the resolution of one crossing, either two circles in D_v *merge* to become one circle in D_u , or, dually, one circle in D_v *splits* to become two circles in D_u . Let $\phi_{v,u}^{\text{op}}: v \rightarrow u$ be the arrow opposite $\phi_{u,v}$.

First, say that two circles $a_1, a_2 \in Z(D_v)$ merge to a circle $a \in Z(D_u)$. Note that the complements $Z(D_v) \setminus \{a_1, a_2\}$ and $Z(D_u) \setminus \{a\}$ are naturally identified. Define $\mathfrak{F}_e(\phi_{v,u}^{\text{op}})$ as the \mathbb{Z} -algebra map

$$\text{Sym}(Z(D_v))/(x^2)_{x \in Z(D_v)} \rightarrow \text{Sym}(Z(D_u))/(x^2)_{x \in Z(D_u)}$$

determined by sending a_1, a_2 to a , and sending other circles by the identity.

Next, say that one circle $a \in Z(D_v)$ splits to circles $a_1, a_2 \in Z(D_u)$. Define

$$\mathfrak{F}_e(\phi_{v,u}^{\text{op}})(x) = (a_1 + a_2)x$$

where we have used the natural identification of $Z(D_v) \setminus \{a\}$ with $Z(D_u) \setminus \{a_1, a_2\}$. One readily checks that, with these definitions, \mathfrak{F}_e defines a functor $(\underline{2}^n)^{\text{op}} \rightarrow \mathbb{Z}\text{-Mod}$.

Gradings

There are two gradings associated to the Khovanov complex: first, there is the *homological grading* gr_h , and an additional *quantum grading* gr_q that allows for decategorification to the Jones polynomial.

Let D be a diagram for an oriented link L , n the number of crossings in D , and n_+ and n_- the number of positive and negative crossings (where a negative crossing is locally \nearrow_{\searrow}) in D , respectively. Let $x = a_1 \dots a_\ell \in \mathfrak{F}_e(D_u)$ (where $a_i \in Z(D_u)$); then the gradings of x are given by

$$\text{gr}_h(x) = |v| - n_-, \quad \text{gr}_q(x) = |Z(D_v)| - 2\ell + |v| + n_+ - 2n_-.$$

Note that the morphisms $\mathfrak{F}_e(\phi_{v,u}^{\text{op}})$ increase homological grading by 1 and preserve quantum grading. In particular, we can regard

$$\mathfrak{F}_e(\phi_{v,u}^{\text{op}}) \rightarrow \mathbb{Z}\text{-gMod}$$

where $\mathbb{Z}\text{-gMod}$ is the category of graded \mathbb{Z} -modules. We write \mathfrak{F}_e^j for the functor taking $(\underline{2}^n)^{\text{op}}$ to the j -graded component of \mathfrak{F}_e .

3.2.3 Homology from functors

Khovanov homology is defined from \mathfrak{F}_e as follows. Let

$$Kc(L) = \bigoplus_{v \in \underline{2}^n} \mathfrak{F}_e(v), \quad \partial_{Kh} = \sum_{v \geq_1 w} (-1)^{s_{v,w}} \mathfrak{F}_e(\phi_{w,v}^{\text{op}}).$$







Here s is the standard sign assignment from Definition 3.7. The chain homotopy type of the resulting complex is an invariant of the oriented link L , [Kho00, Theorem 1]. Note that $Kc(L)$ decomposes, over quantum grading, as a chain complex $Kc(L) = Kc^j(L)$.

The resulting homology $Kh^{i,j}(L) = H^i(Kc^j(L))$ is the *Khovanov homology* of L .

3.2.4 Odd Khovanov homology Kh_o

Odd Khovanov homology, introduced in [ORSz13], is structurally very similar to even Khovanov homology, but instead uses exterior algebra operations to define the differential, introducing signs to the differential within edges. We will view odd Khovanov homology as coming from a functor

$$\mathfrak{F}_o : (\underline{2}^n)^{\text{op}} \longrightarrow \mathbb{Z}\text{-Mod}$$

In order to define odd Khovanov homology from a link diagram D with n ordered crossings, we further equip D with an *orientation of crossings*, which is a choice of an arrow at each crossing. Note that an orientation of the link can be used to acquire an orientation of crossings. The resolution of a diagram D with an orientation of crossings assigns to $v \in \underline{2}^n$ a collection of embedded circles, along with embedded oriented arcs joining the circles. That is, locally the 0-resolution of  (respectively, ) is  (respectively, ) and the 1-resolution is  (respectively, .

For objects $v \in \underline{2}^n$, set $\mathfrak{F}_o(v) = \Lambda(Z(D_v))$, the exterior algebra, over \mathbb{Z} , on the set of symbols $Z(D_v)$. This can be identified with $\mathfrak{F}_e(v)$, but the identification is not canonical. To define \mathfrak{F}_o , we start with an auxiliary assignment \mathfrak{F}'_o (with the same objects) defined on edges $u \geq_1 v$; the functor \mathfrak{F}_o is obtained by changing suitable signs of \mathfrak{F}'_o . We will call \mathfrak{F}'_o the *projective odd Khovanov functor*.

For $u \geq_1 v$, so that circles $a_1, a_2 \in Z(D_v)$ merge to a circle $a \in Z(D_u)$, set $\mathfrak{F}'_o(\phi_{v,u}^{\text{op}})$ to be the \mathbb{Z} -algebra map $\Lambda(Z(D_v)) \rightarrow \Lambda(Z(D_u))$ determined by sending $a_1, a_2 \rightarrow a$ and by identifying the other generators.

For $u \geq_1 v$, so that a circle $a \in Z(D_v)$ splits into circles $a_1, a_2 \in Z(D_u)$, and so that

the arc in D_u points from a_1 to a_2 , set

$$\mathfrak{F}'_o(\phi_{v,u}^{\text{op}})(x) = (a_1 - a_2)x$$

where we view $\Lambda(Z(D_v))$ as a subalgebra of $\Lambda(Z(D_u))$ by sending a to either a_1 or a_2 and identifying the other generators (it is easy to see that \mathfrak{F}'_o does not depend on the choice of where a is sent). It will be convenient later to have the following terminology from [LS14]:

Definition 3.8 (Definition 2.1 [LS14]). A *resolution configuration* C is a pair $(Z(C), A(C))$ where $Z(C)$ is a collection of pairwise-disjoint embedded circles in S^2 , and $A(C)$ is a totally ordered collection of arcs embedded in S^2 with $A(C) \cap Z(C) = \partial A(C)$. The number of arcs will be called the *index* of a resolution configuration.

An *odd* resolution configuration will be such as above, but where the arcs are oriented.

The assignment \mathfrak{F}'_o on the edges of $(\underline{2}^n)^{\text{op}}$ commutes *up to a sign* along 2-dimensional faces. We can adjust \mathfrak{F}'_o to give a genuine functor from the cube category, as follows.

The two-dimensional odd resolution configurations can be divided into four categories as follows (with unoriented arcs being orientable in either direction).

$$\begin{aligned}
A : & \quad \text{---}\ominus\text{---}\ominus, \text{---}\bigcirc\text{---}\bigcirc, \text{---}\bigcirc\text{---}\bigcirc, \text{---}\bigcirc\text{---}\bigcirc. \\
C : & \quad \text{---}\bigcirc\text{---}\bigcirc\text{---}\bigcirc, \text{---}\bigcirc\text{---}\bigcirc, \text{---}\bigcirc\text{---}\bigcirc, \text{---}\bigcirc\text{---}\bigcirc, \text{---}\bigcirc\text{---}\bigcirc, \text{---}\bigcirc\text{---}\bigcirc, \text{---}\bigcirc\text{---}\bigcirc. \\
X : & \quad \text{---}\bigcirc\text{---}\bigcirc. \\
Y : & \quad \text{---}\bigcirc\text{---}\bigcirc.
\end{aligned} \tag{3.9}$$

For a link diagram D and $u \geq_i w \in \underline{2}^n$, we write $D_{u,w}$ for the resolution configuration obtained by performing the w -resolution, and then drawing the i arcs corresponding to the difference between u and w .

Note that \mathfrak{F}'_o commutes on faces of type C, and anticommutes on faces of type A.

Meanwhile, \mathfrak{F}'_o both commutes and anticommutes on faces of type X and type Y (that is, $\mathfrak{F}'_o(\phi_{v,u}^{\text{op}})\mathfrak{F}'_o(\phi_{w,v}^{\text{op}}) = 0$ on faces of type X and type Y). For later reference, we call type X and type Y odd resolution configurations (as well as their underlying resolution configurations) *ladybug configurations*.

We can then define *obstruction cocycles* $\Omega(D) \in C_{\text{cell}}^2([0, 1]^n; \mathbb{Z}_2)$ as follows ($\mathbb{Z}_2 = \{1, -1\}$ will be written multiplicatively). Define the type X (resp. type Y) obstruction cocycle $\Omega(D)^X \in C_{\text{cell}}^2([0, 1]^n; \mathbb{Z}_2)$ (resp. $\Omega(D)^Y$) by setting $\Omega(D)_{u,w}^X = -1$ on faces of type A and type X (resp. type A and type Y), and $\Omega(D)_{u,w}^X = 1$ on faces of type C and type Y (resp. type C and type X). In the sequel we will usually omit the superscript from $\Omega(D)^X$, and usually we will work with the type X obstruction cocycle.

Note that the obstruction cocycle cannot *a priori* be determined from the projective functor $\mathfrak{F}'_o: (\underline{2}^n)^{\text{op}} \rightarrow \mathbb{Z}\text{-Mod}$ itself; the value $\Omega(D)_{u,w}$ on faces $u \geq_2 w \in \underline{2}^n$ so that $\mathfrak{F}'_o(\phi_{v,u}^{\text{op}})\mathfrak{F}'_o(\phi_{w,v}^{\text{op}}) \neq 0$ is determined by \mathfrak{F}'_o , but for faces with $\mathfrak{F}'_o(\phi_{v,u}^{\text{op}})\mathfrak{F}'_o(\phi_{w,v}^{\text{op}}) = 0$, we need the type of $D_{u,w}$ to specify $\Omega(D)_{u,w}$.

It is shown in [ORSz13] that $\Omega(D)$ (for either type) is a cocycle, and so also a coboundary, since $H^2(C_{\text{cell}}([0, 1]^n; \mathbb{Z}_2)) = 0$. That is, there exists some *edge-assignment* $\epsilon \in C_{\text{cell}}^1([0, 1]^n; \mathbb{Z}_2)$ such that $\delta\epsilon = \Omega(D)$, where δ denotes the coboundary of $C_{\text{cell}}([0, 1]^n; \mathbb{Z}_2)$. Moreover, $H^1(C_{\text{cell}}([0, 1]^n; \mathbb{Z}_2)) = 0$, so ϵ is well-defined up to multiplication by the 0-cocycle taking value -1 on all vertices of $[0, 1]^n$.

We define

$$\mathfrak{F}_o(\phi_{v,u}^{\text{op}}) = \epsilon_{u,v}\mathfrak{F}'_o(\phi_{v,u}^{\text{op}}).$$

By definition of ϵ , \mathfrak{F}_o defines a functor from the opposite cube category $(\underline{2}^n)^{\text{op}} \rightarrow \mathbb{Z}\text{-gMod}$. Although the identification of $\mathfrak{F}_o(D_u)$ and $\mathfrak{F}_e(D_u)$ is noncanonical, all choices result in the same grading on $\mathfrak{F}_o(D_u)$. Moreover, it is clear that the arrows $\mathfrak{F}_o(\phi)$ respect q -grading and increase h -grading by 1.

Odd Khovanov homology is constructed from this functor via

$$Kc_o(L) = \bigoplus_{v \in \underline{2}^n} \mathfrak{F}_o(v), \quad \partial_{Kh_o} = \sum_{v \geqslant_1 w} (-1)^{s_{v,w}} \mathfrak{F}_o(\phi_{w,v}^{\text{op}}).$$

The homology $H^i(Kc_o^j, \partial_{Kh_o}) = Kh_o^{i,j}(L)$ is called the *odd Khovanov homology* of L , and its isomorphism class is an invariant of the isotopy class of the oriented link L , [ORSz13]. We will write $Kh_o^j(L)$ for the sum $\bigoplus_i Kh_o^{i,j}(L)$, and similarly for even Khovanov homology.

We will also need to fix bases for the various \mathbb{Z} -modules considered above. For the even case, a natural set of generators is given by elements $a_1 \otimes \cdots \otimes a_k \in \text{Sym}(Z(D_v))/(x^2)_{x \in Z(D_v)}$ where each $a_i \in Z(D_v)$ is distinct. For the odd case, say we have fixed an orientation of crossings and an edge assignment. In order to choose a basis, we fix at every vertex $v \in \underline{2}^n$ a total ordering $>$ on the set $Z(D_v)$. The set

$$Kg(v) = \{a_1 \otimes \cdots \otimes a_k : a_i \in Z(D_v), a_1 > \cdots > a_k\}$$

is called the set of *Khovanov generators at v* .

3.2.5 Annular filtrations

We call a link $L \subset (\mathbb{R}^2 - \{0\}) \times [0, 1]$ an *annular link*; in this section we recall the definition of the annular and odd annular Khovanov homologies of annular links. The former is first defined by [APS06], the latter is a generalization of their construction, first appearing in [GW18].

It is convenient to think of annular links as drawn on $S^2 = \mathbb{R}^2 \cup \{\infty\}$ with two basepoints, with \mathbb{X} at the origin and \mathbb{O} at ∞ . The presence of these basepoints filters both the even and odd Khovanov complexes by a filtration grading gr_k , and the associated graded objects are the *annular Khovanov* and the *odd annular Khovanov* complexes. We will denote their homologies by AKh and AKh_o , respectively.

Fix an annular link diagram D . To obtain the annular ‘ (k) -grading,’ we choose an

oriented arc γ from \mathbb{X} to \mathbb{O} that misses all crossings of D ; the resulting grading will be independent of the choice of γ . For each Kauffman state of a resolution D_u , viewed as a monomial $x_{a_1} \dots x_{a_t}$ in the circles $Z(D_u)$, we obtain an orientation of the circles $Z(D_u)$, where the circles x_{a_i} for $i = 1, \dots, t$ are oriented clockwise and the other circles are oriented counterclockwise. View the collection of oriented circles (associated to a Kauffman state) $Z(D_u)$ as an embedded 1-manifold z . The (k) -grading of $x = x_{a_1} \dots x_{a_t}$, written $\text{gr}_k(x)$, is defined by $\text{gr}_k(x) = I(\gamma, z)$, the algebraic intersection number of γ and z .

One can check that the maps $\mathfrak{F}_e(\phi_{v,u}^{\text{op}})$ and $\mathfrak{F}_o(\phi_{v,u}^{\text{op}})$ (whence also the differentials ∂_{Kh} and ∂_{Kh_o}) can only preserve or decrease the (k) -grading. We set $\mathfrak{F}_{\text{Ann}}^{j,k}(v)$ to be the submodule of $\mathfrak{F}_e^j(v)$ concentrated in annular grading k (similarly for $\mathfrak{F}_{\text{Ann}_o}^{j,k}(v)$). Let $\iota_k: \mathfrak{F}_{\text{Ann}}^{j,k}(v) \rightarrow \mathfrak{F}_e^j(v)$ be the natural inclusion, and let $\pi_k: \mathfrak{F}_e^j(v) \rightarrow \mathfrak{F}_{\text{Ann}}^{j,k}(v)$ be the natural projection. We define the morphisms $\mathfrak{F}_{\text{Ann}}^{j,k}(\phi_{v,u}^{\text{op}})$ to be the (k) -grading preserving part of $\mathfrak{F}_e^j(\phi_{v,u}^{\text{op}})$, that is: $\mathfrak{F}_{\text{Ann}}^{j,k}(\phi_{v,u}^{\text{op}}) = \pi_k \mathfrak{F}_e^j(\phi_{v,u}^{\text{op}}) \iota_k$. The definition for $\mathfrak{F}_{\text{Ann}_o}$ is similar. The even (resp. odd) annular Khovanov functor

$$\mathfrak{F}_{\text{Ann}} : (\underline{2}^n)^{\text{op}} \longrightarrow \mathbb{Z}\text{-Mod},$$

where $\mathfrak{F}_{\text{Ann}} = \bigoplus_{j,k} \mathfrak{F}_{\text{Ann}}^{j,k}$ (resp. $\mathfrak{F}_{\text{Ann}_o}$), is the associated graded object. It will also be convenient to define $\mathfrak{F}'_{\text{Ann}_o}$, the *(odd) annular Khovanov projective functor*, as the associated graded object of \mathfrak{F}'_o .

The even annular Khovanov homology $AKh^{i,j,k}(L) = H^i(AKc^{j,k}(L))$ is defined as the homology of the complex

$$AKc^{j,k}(L) = \bigoplus_{v \in \underline{2}^n} \mathfrak{F}_{\text{Ann}}^{j,k}(v), \quad \partial = \sum_{v \geq_1 w} (-1)^{s_{v,w}} \mathfrak{F}_{\text{Ann}}(\phi_{w,v}^{\text{op}}),$$

and similarly for odd annular Khovanov homology $AKh_o(L)$, as the homology of a

complex $AKc_o(L)$. The isomorphism classes of $AKh(L)$ and $AKh_o(L)$ are invariants of the annular isotopy class of L .

We can also describe the maps $\mathfrak{F}_{\text{Ann}}(\phi_{v,u}^{\text{op}})$ in local pictures. It will be useful later to define *annular resolution configurations* (resp. *odd annular resolution configurations*) as in the definition of resolution configurations, except that we replace with the condition that the embedded circles lie in S^2 with the condition that the embedded circles lie in $S^2 - \{\mathbb{X}, \mathbb{O}\}$. Note that an (odd) annular resolution configuration has a well-defined underlying (odd) resolution configuration. We sometimes abuse notation and refer to any of the above kinds of resolution configurations as just a *configuration*.

There are two types of circles in an annular resolution: we call a circle *nontrivial* if it separates \mathbb{O} and \mathbb{X} and *trivial* otherwise. We associate the \mathbb{Z} -module $\mathbb{V} = \mathbb{Z}[v_+, v_-]$ to nontrivial circles and $\mathbb{W} = \mathbb{Z}[w_+, w_-]$ to trivial circles in order to distinguish the two types of tensor components appearing in $\mathfrak{F}_{\text{Ann}}(v)$. Note that an elementary cobordism in the annulus corresponds to one of six situations, the isotopy classes of index-1 annular resolution configurations. Figure 3.1 shows these elementary cobordisms. For an elementary cobordism $S: D_v \rightarrow D_u$, we call a circle x in $Z(D_v)$ or $Z(D_u)$ active if the component of S containing x is not homeomorphic to a cylinder, otherwise we call x a passive circle. The maps $\mathfrak{F}_{\text{Ann}}(\phi_{v,u}^{\text{op}})$ (and $\mathfrak{F}_{\text{Ann}_o}(\phi_{v,u}^{\text{op}})$) are obtained from the maps in Figure 3.1 (and their split map duals) by tensoring with the identity map on generators corresponding to passive circles.

There is another grading gr_{j_1} special to the annular case that we are tempted to call the *annular quantum grading*, as it appears to be more relevant in annular Khovanov homology than the quantum grading, first introduced in [GLW18] as the ‘filtration-adjusted quantum grading.’ It is defined by $\text{gr}_{j_1} = \text{gr}_q - \text{gr}_k$, and will play an important role when we study the Khovanov complexes for periodic links.

Given an annular link diagram, the Khovanov generators $Kg(v)$ inherit a well-defined

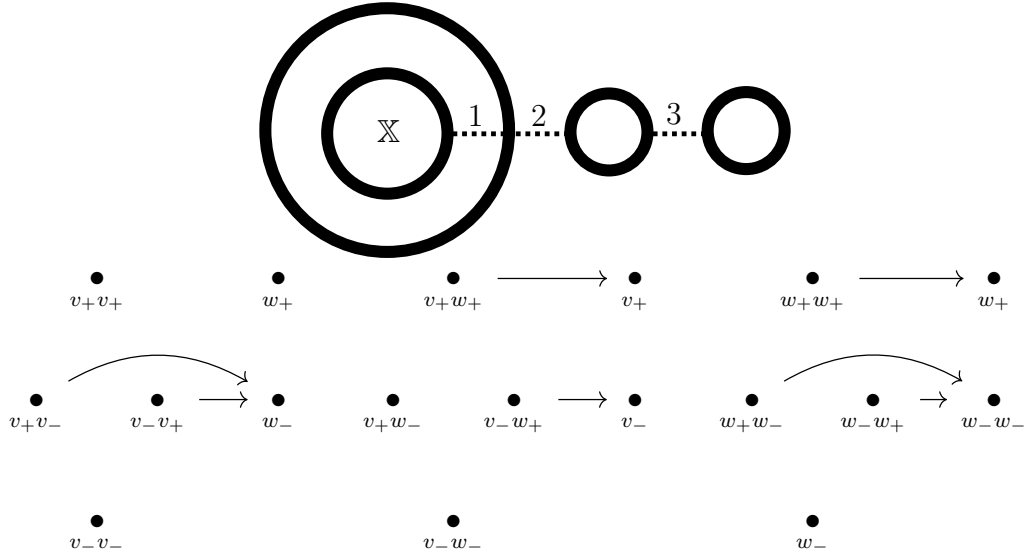


Figure 3.1: Elementary cobordisms in the annulus. Surgering along dotted arc ‘1’ merges two nontrivial circles into a trivial one, corresponding to the merge map $\mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{W}$. Similarly, surgering along dotted arc ‘2’ merges a nontrivial circle with a trivial one, and surgering along dotted arc ‘3’ merges two trivial circles. The reverse surgeries yield the other index-1 annular resolution configurations.

(k) -grading, and we will write $Kg^{j,k}(v)$ for the Khovanov generators at $v \in \underline{2}^n$ with $\text{gr}_q = j$ and $\text{gr}_k = k$.

3.2.6 Periodic links

Here we review some facts about periodic links. The definition is motivated by the resolution [Wal69], [MB84] of the Smith Conjecture, which states that the fixed-point set of any action by \mathbb{Z}_p by orientation-preserving diffeomorphisms of S^3 is the empty set, two points, or an unknotted circle.

A link $\tilde{L} \subset S^3$ is p -periodic if there is an orientation-preserving \mathbb{Z}_p -action ψ on the pair (S^3, \tilde{L}) such that the fixed-point set is an unknot \tilde{U} disjoint from \tilde{L} (often, we will confound notation, and write ψ for a generator of this action). The image of \tilde{L} under the quotient map $S^3 \rightarrow S^3/\psi$ is called the *quotient link*, and is denoted L . We always assume $p > 1$. Two p -periodic links are considered equivalent if there is an *equivariant*

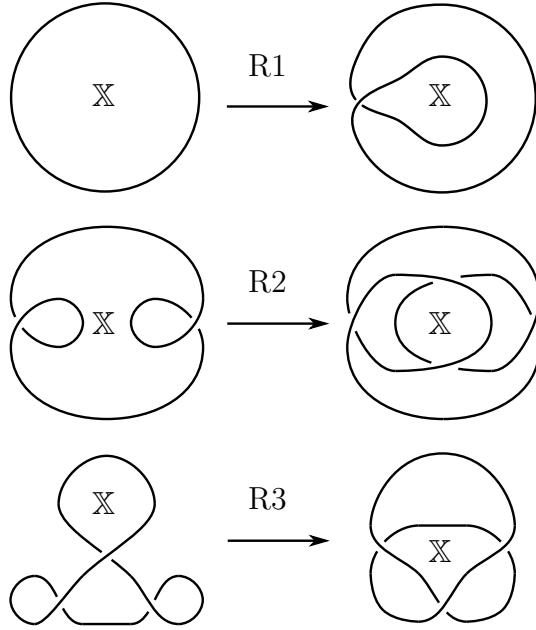


Figure 3.2: Examples where non-annular Reidemeister moves change the isotopy class of the periodic link. The figures in left column show the quotient link L before the Reidemeister move; in each case, the periodic link \tilde{L} is annularly isotopic to the unknot which links once with the axis \tilde{U} . The right column shows the quotient link after the Reidemeister move. For the R1 and R3 cases, \tilde{L} is the torus knot (or link) $T_{2,p}$. For the R2 case, \tilde{L} is the closure of of the braid $(\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1})^p$; in the case $p = 2$, this is the Figure 8 knot.

(ambient) isotopy relating them [Pol15].

Definition 3.10. Two p -periodic links (\tilde{L}_0, ψ_0) and (\tilde{L}_1, ψ_1) are *equivariantly isotopic* if there is an ambient isotopy $\phi_t : S^1 \times [0, 1] \rightarrow S^3$, $0 \leq t \leq 1$, and a homotopy of \mathbb{Z}_p -actions $\psi_t : \mathbb{Z}_p \times S^3 \times [0, 1] \rightarrow S^3$ extending the actions ψ_0, ψ_1 and so that $\phi_i = \tilde{L}_i$ for $i = 0, 1$, and so that ϕ_t is ψ_t -equivariant.

Observe that if we remove the fixed-point set at each time t , such an isotopy can be viewed as an ambient isotopy in the solid torus. By quotienting by the action of ψ_t at each time, we see that this isotopy is a lift of an isotopy from L_0 to L_1 .

A cyclic group can act on a link in distinct ways, and different cyclic groups can act on the same link, so a p -periodic link is defined to be a pair (\tilde{L}, ψ) . Since ψ specifies the unknotted axis \tilde{U} , periodic links are inherently annular.

With this in mind, observe that \tilde{L} has a diagram \tilde{D} on the annulus, where the unknotted axis is viewed as the z -axis $\cup \infty$ and is projected to the basepoint \mathbb{X} , and the induced \mathbb{Z}_p -action on \tilde{D} (also denoted ψ) is simply counterclockwise rotation about \mathbb{X} by $2\pi/p$. We will call such a planar diagram a *periodic diagram* of the periodic link \tilde{L} . Then $D = \tilde{D}/\psi$ is a diagram for the quotient link L . We will usually assume that all of our diagrams for p -periodic links are periodic diagrams.

Note also that given an annular diagram D , we can form a p -periodic link diagram \tilde{D} , called the *p-cover* of D , by taking p copies $\{D_i\}_{i=1,\dots,p}$ of D cut along an arc γ as in the definition of annular Khovanov homology, and gluing (reversing orientation on the boundary) D_i to D_{i+1} along one boundary component of the cut diagram (with subscripts interpreted cyclically).

With this notion of periodic diagrams, given p -periodic diagrams \tilde{D}_1 and \tilde{D}_2 , they represent the same periodic link if and only if they are related by equivariant isotopies and *equivariant Reidemeister moves*, which are the lift of Reidemeister moves on the quotient diagrams D_1, D_2 (see [Pol15]). See Figure 3.9 for instance.

Remark 3.11. In particular, equivariant Reidemeister moves do not interact with the basepoint \mathbb{X} in the diagram. Figure 3.2 provides examples showing why we should expect equivariant moves to be annular: moves that do interact with the basepoint can change the isotopy class of the periodic link in S^3 .

For bookkeeping purposes, we introduce the notation that $\tilde{\cdot}$ generally means ‘lift of’, as well as the following rules. Given an ordering of crossings of a diagram D , we obtain an ordering of crossings upstairs as follows. Recall that in the definition of annular Khovanov homology we relied on an arc γ . As the quotient of a periodic diagram \tilde{D} , the diagram D is naturally an annular diagram, and we fix some arc γ as in the definition of annular Khovanov homology. Lift it to some arc $\tilde{\gamma}$ on \tilde{D} . We divide the plane containing \tilde{D} into *sectors*, that is, the connected components of $\mathbb{R}^2 - \mathbb{Z}_p \tilde{\gamma}$, where $\mathbb{Z}_p \tilde{\gamma}$ denotes the

orbit of $\tilde{\gamma}$ under \mathbb{Z}_p . The sectors are labeled S_1, \dots, S_p , where S_i is the sector between $\psi^{i-1}\tilde{\gamma}$ and $\psi^i\tilde{\gamma}$. The crossings of \tilde{D} are ordered by requiring that the first n crossings are just those contained in S_1 , ordered according to their ordering in the quotient, the next n are the crossings of S_2 , and so on. From now on, unless otherwise stated, given an annular diagram D with ordered crossings, we will assume its p -cover \tilde{D} has this ordering of crossings.

There is also an induced action ψ on the Khovanov generators, by acting by rotation on resolution diagrams. That is, \mathbb{Z}_p acts on $\oplus_v \mathfrak{F}_e(v)$ by sending a Kauffman state $x_1 \dots x_t$ to $y_1 \dots y_t$, where y_i is the result of rotation on x_i . For the above ordering of the crossings of \tilde{D} and D , this action lies over the action of \mathbb{Z}_p on $(\mathbb{Z}^n)^p$ by cyclic permutation. Call a Khovanov generator an *invariant generator* if it is invariant under the action of \mathbb{Z}_p . Meanwhile, \mathbb{Z}_p does act by bijections on the set $Kg(\tilde{D})$, but one can say somewhat more. That is, \mathbb{Z}_p may send (odd) Khovanov generators to \pm -multiples of odd Khovanov generators. Let a *signed bijection* $X: S_1 \rightarrow S_2$ between two finite sets S_1, S_2 be a bijection along with a ‘sign’ map $\sigma: S_1 \rightarrow \mathbb{Z}_2$ (really, we should view X as a correspondence between S_1 and S_2 along with a ‘sign’ map $\sigma: X \rightarrow \mathbb{Z}_2$. See Section 3.3). Then the generator ψ of \mathbb{Z}_p acts by signed bijections, $Kg(u) \rightarrow Kg(\psi u)$, where the sign of $x \in Kg(u)$ records the sign of the generator $\psi(x)$ as a Khovanov generator of $\mathfrak{F}_o(\psi u)$. We write $Kg(\tilde{D})^{\mathbb{Z}_p}$ for the set of invariant Khovanov generators (where invariant just means invariant under the \mathbb{Z}_p -action, and does not involve the sign map of the \mathbb{Z}_p -action).

We conclude this section by discussing the relationship between generators in $Kc(D)$ and their lifts in $Kc(\tilde{D})$; in particular, the relationship between their gradings explains the role annular filtrations play in localization of Khovanov homology.

Proposition 3.12 (cf. Proposition 29, [Zha18]). *There is a bijective correspondence between the generators of $Kc(D)$ and the invariant generators of $Kc(\tilde{D})$, given by $x \mapsto \tilde{x}$,*

such that

$$\mathrm{gr}_k(\tilde{x}) = \mathrm{gr}_k(x), \quad \mathrm{gr}_h(\tilde{x}) = p\mathrm{gr}_h(x), \quad \text{and} \quad \mathrm{gr}_q(\tilde{x}) = p\mathrm{gr}_q(x) - (p-1)\mathrm{gr}_k(x).$$

In particular, this implies $\mathrm{gr}_{j_1}(\tilde{x}) = p\mathrm{gr}_{j_1}(x)$.

Proof. Note that $\tilde{n}_+ = pn_+$, $\tilde{n}_- = pn_-$, and $|\tilde{u}| = p|u|$. Let $x \in Kc(D)$ be a generator lying at vertex $u \in \underline{2}^n$. Suppose D_u has α nontrivial circles labeled v_+ , β nontrivial circles labeled v_- , γ trivial circles labeled w_+ , and δ trivial circles labeled w_- . Up to permuting the tensor factors around, we may write $x = v_+^\alpha v_-^\beta w_+^\gamma w_-^\delta$.

Let S be a circle in D_u . If S is nontrivial, then its lift in $D_{\tilde{u}}$ consists of a single equivariant nontrivial circle. On the other hand, if S is trivial, then its lift consists of p identical copies of a nontrivial circle. In light of this observation, we may write $\tilde{x} = v_+^\alpha v_-^\beta w_+^{p\gamma} w_-^{p\delta}$. Now we may compute the following.

$$\begin{aligned} \mathrm{gr}_k(\tilde{x}) &= \alpha - \beta = \mathrm{gr}_k(x) \\ \mathrm{gr}_h(\tilde{x}) &= |\tilde{u}| - \tilde{n}_- = p|u| - pn_- = p\mathrm{gr}_h(x) \\ \mathrm{gr}_q(\tilde{x}) &= |\tilde{u}| + \alpha - \beta + p\gamma - p\delta + \tilde{n}_- - 2\tilde{n}_+ \\ &= p|u| + p(\alpha - \beta + \gamma - \delta) - (p-1)(\alpha - \beta) + p(n_- - 2n_+) \\ &= p\mathrm{gr}_q(x) - (p-1)\mathrm{gr}_k(x). \end{aligned}$$

The gr_{j_1} relationship follows directly. □

Moreover, as a matter of conventions, Proposition 3.12 extends to a signed bijective correspondence $Kg(D) \rightarrow Kg(\tilde{D})^{\mathbb{Z}_p}$, when the order of circles upstairs is chosen to satisfy the following. First, if circles $a_1, a_2 \in Z(D_u)$ satisfy $a_1 < a_2$, then any circles over them, say \tilde{a}_1 and \tilde{a}_2 , satisfy $\tilde{a}_1 < \tilde{a}_2$. Further, for $a \in Z(D_u)$, let \tilde{a}_1 be the circle upstairs that is closest to $\tilde{\gamma}$, proceeding counterclockwise from $\tilde{\gamma}$, and where we require that \tilde{a}_1 not

intersect $\tilde{\gamma}$. For nontrivial circles, which necessarily intersect $\tilde{\gamma}$, there is no ambiguity.) Define $\tilde{a}_i = \psi^i \tilde{a}$ for $i \leq p$. We require $a_1 < \dots < a_p$. The bijection $Kg(D) \rightarrow Kg(\tilde{D})^{\mathbb{Z}_p}$ takes nontrivial circles to nontrivial circles, and takes a trivial circle a to $\tilde{a}_1 \dots \tilde{a}_p$.

3.3 Burnside categories and functors

In this section we recall the machinery of Burnside functors from [LLS],[LLS17a]. We will also record a slight generalization of the signed Burnside functors of [SSS18]. The sections 3.3.1-3.3.3 are essentially a review of material from [LLS]-[SSS18]. In section 3.3.4, we introduce external actions on Burnside functors and prove basic properties. The rest of the section consists of generalizing notions of [LLS] to Burnside functors with external action.

3.3.1 The Burnside category

Given finite sets X and Y , a correspondence from X to Y is a triple (A, s, t) for a finite set A , where s, t are set maps $s: A \rightarrow X$ and $t: A \rightarrow Y$; s and t are called the *source* and *target* maps, respectively. The correspondence (A, s, t) is depicted:

$$\begin{array}{ccc} & A & \\ s_A \swarrow & & \searrow t_A \\ X & & Y \end{array}$$

For correspondences (A, s_A, t_A) and (B, s_B, t_B) from X to Y and Y to Z , respectively, define the composition $(B, s_B, t_B) \circ (A, s_A, t_A)$ to be the correspondence (C, s, t) from X to Z given by the fiber product $C = B \times_Y A = \{(b, a) \in B \times A \mid t(a) = s(b)\}$ with source and target maps $s(b, a) = s_A(a)$ and $t(b, a) = t_B(b)$. There is also the identity correspondence from a set X to itself, i.e., $(X, \text{Id}_X, \text{Id}_X)$ from X to X . Given correspondences (A, s_A, t_A) , (B, s_B, t_B) from X to Y , a morphism of correspondences

(A, s_A, t_A) to (B, s_B, t_B) is a bijection $f: A \rightarrow B$ commuting with the source and target maps. There is also the identity morphism from a correspondence to itself.

Composition (of set maps) gives the set of correspondences from X to Y the structure of a category. Define the *Burnside category* \mathcal{B} to be the weak 2-category whose objects are finite sets, morphisms are finite correspondences, and 2-morphisms are maps of correspondences.

Recall that in a weak 2-category, that arrows need only be associative up to an equivalence, and similarly the identity axiom holds only after composing with a 2-morphism. To be explicit, for finite sets X, Y and (A, s, t) a correspondence from X to Y , neither $(Y, \text{Id}_Y, \text{Id}_Y) \circ (A, s, t)$, nor $(A, s, t) \circ (X, \text{Id}_X, \text{Id}_X)$, equals (A, s, t) . Rather, there are natural 2-morphisms, left and right unitors,

$$\lambda: Y \times_Y A \rightarrow A, \quad \rho: A \times_X X \rightarrow A$$

given by $\lambda(y, a) = a$ and $\rho(a, x) = a$. Further, the composition $C \circ (B \circ A)$, for A from W to X , B from X to Y , and C from Y to Z , is not identical to $(C \circ B) \circ A$, rather there is an associator

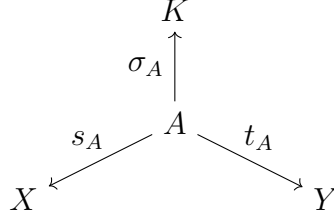
$$\alpha: (C \times_Y B) \times_X A \rightarrow C \times_Y (B \times_X A)$$

given by $\alpha((c, b), a) = (c, (b, a))$. The categories to follow are slight variations of this one.

3.3.2 Decorated Burnside categories

Fix a group K (for our purposes, usually the cyclic group $\mathbb{Z}_2 = \{1, -1\}$). Given finite sets X and Y , a *decorated correspondence* is a correspondence (A, s_A, t_A) equipped with a map $\sigma_A: A \rightarrow K$, regarded as a tuple (A, s_A, t_A, σ_A) ; we call σ_A the “decoration” of

the correspondence (or the “sign” if $K = \mathbb{Z}_2$):



In the sequel we often write ‘correspondence’ for ‘decorated correspondence,’ where it will not cause any confusion. We define a composition $(B, s_B, t_B, \sigma_B) \circ (A, s_A, t_A, \sigma_A)$ of decorated correspondences (A, s_A, t_A, σ_A) from X to Y , and (B, s_B, t_B, σ_B) from Y to Z by (C, s, t, σ) , where (C, s, t) is the composition $(B, s_B, t_B) \circ (A, s_A, t_A)$ and $\sigma(b, a) = \sigma_B(b)\sigma_A(a)$. Also, we define the identity correspondence by $(X, \text{Id}_X, \text{Id}_X, 1)$ (i.e., the identity correspondence takes value 1 on all elements).

We define maps of decorated correspondences $f: (A, s_A, t_A, \sigma_A) \rightarrow (B, s_B, t_B, \sigma_B)$ to be morphisms of correspondences $f: (A, s_A, t_A) \rightarrow (B, s_B, t_B)$ such that $\sigma_B \circ f = \sigma_A$. We may then define the K -Burnside category \mathcal{B}_K to be the weak 2-category with objects finite sets, morphisms given by decorated correspondences, and 2-morphisms given by maps of decorated correspondences. The structure maps λ, ρ, α of §3.3.1 are easily seen to respect the decoration, confirming that \mathcal{B}_K is indeed a weak 2-category. There is a forgetful 2-functor $\mathcal{F}: \mathcal{B}_K \rightarrow \mathcal{B}$ which forgets decorations. We will usually refer to such 2-functors simply as functors.

For a homomorphism $\mathfrak{d}: K \rightarrow \mathbb{Z}_2$ we define a functor $\mathcal{A}_{\mathfrak{d}}: \mathcal{B}_K \rightarrow \mathbb{Z}\text{-Mod}$ by sending a set $X \in \mathcal{B}_K$ to the free abelian group generated by X , denoted $\mathcal{A}_{\mathfrak{d}}(X)$. For a decorated correspondence $\phi = (A, s, t, \sigma)$ from X to Y , we define $\mathcal{A}_{\mathfrak{d}}(\phi): \mathcal{A}_{\mathfrak{d}}(X) \rightarrow \mathcal{A}_{\mathfrak{d}}(Y)$ by

$$\mathcal{A}_{\mathfrak{d}}(\phi)(x) = \sum_{a \in A \mid s(a)=x} \mathfrak{d}(\sigma(a))t(a) \tag{3.13}$$

for elements $x \in X$, extended linearly over \mathbb{Z} . We often suppress \mathfrak{d} from the notation.

3.3.3 Functors to Burnside categories

We now consider functors from the cube category $\underline{2}^n$ to the Burnside categories thus far introduced. The functors $F: \underline{2}^n \rightarrow \mathcal{B}_K$ we consider will be strictly unitary 2-functors; that is, they will consist of the following data:

1. For each vertex v of $\underline{2}^n$, an object $F(v)$ of \mathcal{B}_K .
2. For any $u \geq v$, a 1-morphism $F(\phi_{u,v})$ in \mathcal{B}_K from $F(u)$ to $F(v)$, such that $F(\phi_{u,u})$ is the identity morphism $\text{Id}_{F(u)}$.
3. Finally, for any $u \geq v \geq w$, a 2-morphism $F_{u,v,w}$ in \mathcal{B}_K from $F(\phi_{v,w}) \circ F(\phi_{u,v})$ to $F(\phi_{u,w})$ that agrees with λ (respectively, ρ) when $v = w$ (respectively, $u = v$), and that satisfies, for any $u \geq v \geq w \geq z$,

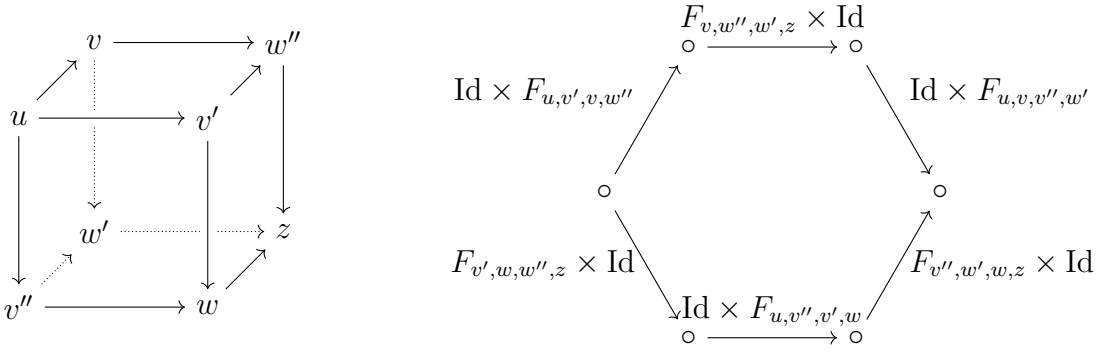
$$F_{u,w,z} \circ_2 (\text{Id} \circ F_{u,v,w}) = F_{u,v,z} \circ_2 (F_{v,w,z} \circ \text{Id})$$

(with \circ denoting composition of 1-morphisms and \circ_2 denoting composition of 2-morphisms; and we have suppressed the associator α).

We will usually use the characterization of these functors in the lemma to follow.

Lemma 3.14. *[Lemma 3.2 [SSS18]] Consider the data of objects $F(v)$ for $v \in \underline{2}^n$, a collection of 1-morphisms $F(\phi_{v,w})$, in \mathcal{B}_K , for edges $v \geq_1 w$, and 2-morphisms $F_{u,v,v',w}: F(\phi_{v,w}) \circ F(\phi_{u,v}) \rightarrow F(\phi_{v',w}) \circ F(\phi_{u,v'})$ for each 2-dimensional face described by $u \geq_1 v, v' \geq_1 w$, such that the following compatibility conditions are satisfied:*

1. *For any 2-dimensional face u, v, v', w as above, $F_{u,v,v',w} = F_{u,v',v,w}^{-1}$;*
2. *For any 3d face in $\underline{2}^n$ on the left, the hexagon on the right commutes:*



This collection of data can be extended to a strictly unitary functor $F: \underline{2}^n \rightarrow \mathcal{B}_K$, uniquely up to natural isomorphism, so that $F_{u,v,v',w} = F_{u,v',w}^{-1} \circ_2 F_{u,v,w}$.

Definition 3.15. Given a functor $F: \underline{2}^n \rightarrow \mathcal{B}_K$ and $\mathfrak{d}: K \rightarrow \mathbb{Z}_2$, we construct a chain complex denoted $\text{Tot}_{\mathfrak{d}}(F)$, and called the *totalization* of the functor F . We usually suppress \mathfrak{d} from notation when it is clear. The underlying chain group of $\text{Tot}(F)$ is given by

$$\text{Tot}(F) = \bigoplus_{v \in \underline{2}^n} \mathcal{A}(F(v)).$$

We set the homological grading of the summand $\mathcal{A}(F(v))$ to be $|v|$. The differential is given by defining the components $\partial_{u,v}$ from $\mathcal{A}(F(u))$ to $\mathcal{A}(F(v))$ by

$$\partial_{u,v} = \begin{cases} (-1)^{s_{u,v}} \mathcal{A}(F(\phi_{u,v})) & \text{if } u \geq_1 v \\ 0 & \text{otherwise.} \end{cases}$$

3.3.4 External actions on Burnside functors

We will be especially interested in Burnside functors that admit ‘extra symmetries’, as follows. Throughout, we require that G is a finite group.

By an action of G on a small category \mathcal{C} , we mean a group action ψ of G on $\text{Ob}(\mathcal{C})$, along with an isomorphism of sets $\psi_g: \text{Hom}(x, y) \rightarrow \text{Hom}(\psi_g x, \psi_g y)$ for each $g \in G$, compatible with composition of morphisms in \mathcal{C} and so that $\psi_h \psi_g = \psi_{hg}$. We further

require that the group action preserves identity morphisms. Equivalently, this is the data of functors $\mathbf{F}_g: \mathcal{C} \rightarrow \mathcal{C}$, for $g \in G$, satisfying the appropriate relations.

Definition 3.16. Fix a Burnside functor $F: \mathcal{C} \rightarrow \mathcal{B}_K$, for \mathcal{C} a small category. Say there exists an action of G by ψ on \mathcal{C} . An *external action* ψ on F consists of the following data:

1. A collection of 1-isomorphisms

$$\psi_{g,v}: F(v) \rightarrow F(gv),$$

in \mathcal{B}_K , for all g, v , subject to $\psi_{gh,v} = \psi_{g,hv} \circ \psi_{h,v}$ (where the equality is to be read as there being a fixed choice of 2-morphism between the two sides of the equation, which we suppress from notation).

2. A collection of 2-morphisms

$$\psi_{g,A}: \psi_{g,t(A)} \circ F(A) \rightarrow F(gA) \circ \psi_{g,s(A)}$$

for all $g \in G, A \in \text{Hom}(\mathcal{C})$, with source $s(A)$ and target $t(A)$.

The data are subject to the following conditions:

(E-1) The 2-morphism $\psi_{gh,A}$ is given by the composite:

$$\begin{aligned} \psi_{gh,t(A)} \circ F(A) &= \psi_{g,ht(A)} \psi_{h,t(A)} \circ F(A) \xrightarrow{\psi_{h,A}} \psi_{g,ht(A)} \circ F(hA) \circ \psi_{h,s(A)} \\ &\xrightarrow{\psi_{g,hA}} F(ghA) \circ \psi_{g,hs(A)} \psi_{h,s(A)} = F(ghA) \circ \psi_{gh,s(A)}. \end{aligned}$$

Schematically,

$$\begin{array}{ccc} u \rightarrow hu \rightarrow ghv & & u \rightarrow ghv \\ \downarrow \nearrow \downarrow \nearrow \downarrow & = & \downarrow \nearrow \downarrow \\ v \rightarrow hv \rightarrow ghv & & v \rightarrow ghv \end{array}$$

(E-2) The following pentagon commutes:

$$\begin{array}{ccc} & F(B) \circ \psi_{g,s(B)} \circ F(A) & \\ \psi_{g,B} \circ_2 \text{Id} \nearrow & & \searrow \text{Id} \circ_2 \psi_{g,A} \\ \psi_{g,t(B)} \circ F(B) \circ F(A) & & F(B) \circ F(A) \circ \psi_{g,s(A)} \\ \downarrow \text{Id} \circ_2 \alpha_{A,B} & & \downarrow \alpha_{A,B} \circ_2 \text{Id} \\ \psi_{g,t(B \circ A)} \circ F(B \circ A) & \xrightarrow{\psi_{g,A \circ B}} & F(B \circ A) \circ \psi_{g,s(A \circ B)} \end{array}$$

Schematically, this says:

$$\begin{array}{ccc} u \rightarrow v \rightarrow w & & u \rightarrow w \\ \downarrow \nearrow \downarrow \nearrow \downarrow & = & \downarrow \nearrow \downarrow \\ gu \rightarrow gv \rightarrow gw & & gu \rightarrow gw \end{array}$$

Remark 3.17. An external action on a Burnside functor F can alternatively be described as follows. Let $\tilde{\mathcal{C}}$ denote the category whose objects are those of \mathcal{C} and whose morphisms are given by $\text{Hom}_{\tilde{\mathcal{C}}}(x, y) = \coprod_{g \in G} \text{Hom}_{\mathcal{C}}(x, g^{-1}y)$, where we call the summand associated to $g \in G$ the g -labelled (not decorated) summand. Composition is given by $\text{Hom}_{\mathcal{C}}(y, h^{-1}z) \times \text{Hom}_{\mathcal{C}}(x, g^{-1}y) \rightarrow \text{Hom}_{\mathcal{C}}(x, (hg)^{-1}z)$ by sending $\text{Hom}_{\mathcal{C}}(y, h^{-1}z) \rightarrow \text{Hom}_{\mathcal{C}}(g^{-1}y, g^{-1}h^{-1}z)$ using the functor $\mathbf{F}_{g^{-1}}$, and then using composition of arrows in \mathcal{C} . In particular, this composition law takes the g -labeled component of $\text{Hom}_{\tilde{\mathcal{C}}}(x, y)$ and the h -labeled component of $\text{Hom}_{\tilde{\mathcal{C}}}(y, z)$ to the hg -labeled component of $\text{Hom}_{\tilde{\mathcal{C}}}(x, z)$.

It is a pleasant exercise to show that Burnside functors F with external action are identified with functors $\tilde{\mathcal{C}} \rightarrow \mathcal{B}_K$.

Note also that Definition 3.16 can be substantially simplified in the case $K = \{1\}$.

Remark 3.18. The complex $\text{Tot}(F)$, for $F: \underline{2}^{np} \rightarrow \mathcal{B}_K$ admitting an external \mathbb{Z}_p -action by permutation of the coordinates, admits its own \mathbb{Z}_p -action as follows. For each $u \in \underline{2}^{np}$, let $\tau(u) = (\#\{i \leq n(p-1) \mid u_i = 1\})(\#\{i > n(p-1) \mid u_i = 1\})$. Define $\psi_*: \text{Tot}(F) \rightarrow \text{Tot}(F)$ by, for $x \in F(v)$, setting $\psi_*(x) = \tau(v)\psi(x)$, where $\psi(x)$ refers to the object in $F(\psi(v))$, coming from the external action on F . It is a direct but tedious check to see that ψ_* is a chain map. Moreover, $\psi_*^p = \text{Id}$, giving $\text{Tot}(F)$ the structure of a chain complex with \mathbb{Z}_p -action. We will not need this structure but we try to make note of it in the sequel; see also Section 3.6.6.

Lemma 3.19. *Let \mathbb{Z}_p act on $(\underline{2}^n)^p$ by cyclic permutation of the factors. Consider the data F as in Lemma 3.14 along with the following data:*

1. *A collection of 1-isomorphisms (in \mathcal{B}_K) $\psi_{g,v}: F(v) \rightarrow F(gv)$, for $v \in \underline{2}^{np} = (\underline{2}^n)^p, g \in \mathbb{Z}_p$. We require that these 1-isomorphisms satisfy $\psi_{gh,v} = \psi_{g,hv}\psi_{h,v}$.*
2. *For each $g \in \mathbb{Z}_p$ and pair $u \geq_1 v \in \underline{2}^{np}$, a 2-morphism $\psi_{g,u,v}: \psi_{g,v} \circ F(\phi_{u,v}) \rightarrow F(\phi_{gu,gv}) \circ \psi_{g,u}$.*

Assume that the data satisfies the following conditions:

(E-1 ') For $u \geq_1 v \in \underline{2}^{np}$, we have

$$\psi_{gh,u,v} = (\psi_{g,hu,hv} \circ \text{Id}) \circ_2 (\text{Id} \circ \psi_{h,u,v}),$$

for all g, h . That is, the data $(\psi_{g,v}, \psi_{g,u,v})$ satisfy (E-1) for length 1 morphisms.

(E-2 ') Write $F(\phi_{u,v}) = A_{u,v}$ to ease the notation. For $u \geq_1 v, v' \geq_1 w$, the following

hexagon commutes:

$$\begin{array}{ccc}
& A_{gv,gw} \circ \psi_{g,v} \circ A_{u,v} \rightarrow A_{gv,gw} \circ A_{gu,gv} \circ \psi_{g,u} & \\
& \nearrow & \searrow F_{gu,gv,gv',gw} \circ \text{Id} \\
\psi_{g,w} \circ A_{v,w} \circ A_{u,v} & & A_{gv',gw} \circ A_{gu,gv'} \circ \psi_{g,u} \\
& \searrow \text{Id} \circ F_{u,v,v',w} & \nearrow \\
& \psi_{g,w} \circ A_{v',w} \circ A_{u,v'} \rightarrow A_{gv',gw} \circ \psi_{g,v'} \circ A_{u,v'} &
\end{array}$$

This collection of data extends to a strictly unitary functor $F: (\underline{2}^n)^p \rightarrow \mathcal{B}_K$ admitting an external \mathbb{Z}_p -action, which is unique up to \mathbb{Z}_p -equivariant natural isomorphism, among strictly unitary functors as constructed in Lemma 3.14 admitting an external action extending the data.

Proof. We will need to briefly describe the argument for Lemma 3.14, which is identical to that of Proposition 4.3 [LLS17a]. The functor F constructed in Lemma 3.14 is defined by, for each $\phi_{u,v}$, choosing a sequence $u \geq_1 u_1 \cdots \geq_1 u_{i-1} \geq_1 u_i = v$ and then setting $F(\phi_{u,v}) = F(\phi_{u_{i-1},v}) \circ \cdots \circ F(\phi_{u,u_1})$. For each $u \geq_i v \geq_j w$, one defines a 2-morphism $F_{u,v,w}: F(\phi_{v,w}) \circ F(\phi_{u,v}) \rightarrow F(\phi_{u,w})$ as follows. By construction, we need a bijection of decorated sets

$$(F(\phi_{v_{j-1},w}) \circ \cdots \circ F(\phi_{v,v_1})) \circ (F(\phi_{u_{i-1},v}) \circ \cdots \circ F(\phi_{u,u_1})) \rightarrow F(\phi_{u'_{i+j-1},w}) \cdots \circ F(\phi_{u,u'_1})$$

for some sequences $u \geq_1 u_1 \cdots \geq_1 u_i = v$, $v \geq_1 v_1, \cdots \geq_1 v_j = w$, and $u \geq_1 u'_1 \cdots \geq_1 u'_{i+j} = w$. Such a bijection is obtained by taking a composition of bijections of the form $\text{Id} \circ F_{x,y,y',z} \circ \text{Id}$ as in the statement of Lemma 3.14. The condition (2) of Lemma 3.14 guarantees that the bijection of decorated sets thus constructed is independent of the

choices of the $F_{x,y,y',z}$.

We need to define 2-isomorphisms in \mathcal{B}_K , $\psi_{g,u,v}: \psi_{g,v} \circ A_{u,v} \rightarrow A_{gu,gv} \circ \psi_{g,u}$ for all $u \geq v$ so that (E-1 ') and (E-2 ') hold. Recall that in the construction of F , for each $u \geq_i v$, we selected a sequence $u \geq_1 u_1 \cdots \geq_1 u_i = v$, and set $A_{u,v} = A_{u_{i-1},v} \circ \cdots \circ A_{u,u_1}$. We have a diagram:

$$\begin{array}{ccc} \psi_{g,v} \circ A_{u_{i-1},v} \circ \cdots \circ A_{u,u_1} & \rightarrow & A_{gu_{i-1},gv} \circ \psi_{g,u_{i-1}} \circ \cdots \circ A_{u,u_1} \rightarrow \cdots \rightarrow A_{gu_{i-1},gv} \circ \cdots \circ A_{gu,gu_1} \circ \psi_{g,u} \\ \downarrow & & \downarrow \\ \psi_{g,v} \circ A_{u,v} & \dashrightarrow & A_{gu,gv} \circ \psi_{g,u} \end{array}$$

where the vertical 2-morphisms are given by the construction of F : the left one is part of the definition, and the right one arises from a sequence of bijections of the form $\text{Id} \circ F_{x,y,y',z} \circ \text{Id}$, as in the proof of Lemma 3.14. Although the decomposition of the vertical into the $F_{x,y,y',z}$ is not well-defined, the resulting composite (the vertical 2-morphism in the diagram) is well-defined. We define the action $\psi_{g,u,v}$ to make the diagram commutative.

For checking that (E-1) holds, we draw the following schematic figures, which the determined reader can translate into equations. Let us set up some notation. Say that in the definition of F , we have selected the sequences $u \geq_1 u_1 \geq_1 \cdots \geq_1 u_i = v$ and $hu \geq_1 u'_1 \geq_1 \cdots \geq_1 u'_i = hv$ and $ghu \geq_1 u''_1 \geq_1 \cdots \geq_1 u''_i = ghv$ to define

$A_{u,v}, A_{hu,hv}, A_{ghu,ghv}$, respectively. We need to compare the composite of 2-morphisms:

$$\begin{array}{ccccc}
 u & \xrightarrow{\quad} & hu & & hu & \xrightarrow{\quad} & ghu & & ghu \\
 \downarrow & \nearrow & \downarrow & & \downarrow & \nearrow & \downarrow & & \downarrow \\
 u_1 & \xrightarrow{\quad} & hu_1 & & u'_1 & \xrightarrow{\quad} & gu'_1 & & u''_1 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \vdots & & \vdots & \xrightarrow{\quad \Phi \quad} & \vdots & & \vdots & \xrightarrow{\quad} & \vdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 u_{i-1} & \xrightarrow{\quad} & hu_{i-1} & & u'_{i-1} & \xrightarrow{\quad} & gu'_{i-1} & & u''_{i-1} \\
 \downarrow & \nearrow & \downarrow & & \downarrow & \nearrow & \downarrow & & \downarrow \\
 u_i = v & \xrightarrow{\quad} & hu_i = hv & & u'_i = hv & \rightarrow & ghu_i = ghv & & u''_i = ghv
 \end{array} \tag{3.20}$$

with

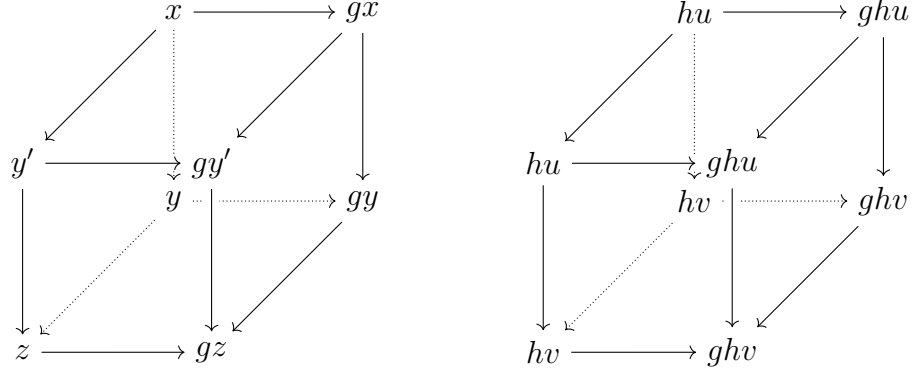
$$\begin{array}{ccccc}
 u & \xrightarrow{\quad} & ghu & & ghu \\
 \downarrow & \nearrow & \downarrow & & \downarrow \\
 u_1 & \xrightarrow{\quad} & ghu_1 & & u''_1 \\
 \downarrow & & \downarrow & & \downarrow \\
 \vdots & & \vdots & \xrightarrow{\quad} & \vdots \\
 \downarrow & & \downarrow & & \downarrow \\
 u_{i-1} & \xrightarrow{\quad} & ghu_{i-1} & & u''_{i-1} \\
 \downarrow & \nearrow & \downarrow & & \downarrow \\
 u_i = v & \rightarrow & ghu_i = ghv & & u''_i = ghv
 \end{array} \tag{3.21}$$

Here the horizontal 2-morphisms come from composing several of the $F_{x,y,y',z}$ maps. We first apply the hypothesis (E-1 ') to express (3.21) as:

$$\begin{array}{ccccccc}
 u & \xrightarrow{\quad} & hu & \xrightarrow{\quad} & ghu & & u \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & \downarrow \\
 u_1 & \xrightarrow{\quad} & hu_1 & \xrightarrow{\quad} & ghu_1 & & u''_1 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \vdots & & \vdots & & \vdots & \xrightarrow{\quad} & \vdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 u_{i-1} & \xrightarrow{\quad} & hu_{i-1} & \xrightarrow{\quad} & ghu_{i-1} & & u''_{i-1} \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & \downarrow \\
 u_i = v & \xrightarrow{\quad} & hu_i = hv & \rightarrow & ghu_i = ghv & & ghv
 \end{array} \tag{3.22}$$

Observe that the first pair of columns of (3.22) is the first 3 columns of (3.20), followed by the 2-morphism Φ^{-1} .

Consider the cubes of 1-morphisms, where $x \geq_1 y, y' \geq_1 z$:



In the left-hand cube, there are two paths of 2-morphisms (coming from 2-morphisms associated to each face of the cube) from $x \rightarrow y \rightarrow gy \rightarrow gz$ to $x \rightarrow y' \rightarrow gy' \rightarrow gz$, which agree by (E-2'). Here the left-hand face is $F_{x,y,y',z}$ and the right-hand face is $F_{gx,gy,gy',gz}$.

Since Φ^{-1} is built as a composite of the $F_{x,y,y',z}$, we have that Φ^{-1} fits into a similar cube ‘of commuting 2-morphisms’, built by composing the small cubes on the left, to form the cube on the right.

The top and bottom faces of the right cube are the identity, while the back face comes from 3.20 and the front face is from 3.22. Another application of (2) from Lemma 3.14 then suffices to verify (E-1).

The proof of (E-2) is proved by substantially similar techniques (but does not require (E-1')), and is left to the reader.

The proof of uniqueness up to natural isomorphism is analogous to the proof that F itself is (nonequivariantly) well-defined up to natural isomorphism.

□

Let H a subgroup of G . For a small category \mathcal{C} with a G -action, let \mathcal{C}^H , called the H -fixed-point category, be the subcategory of \mathcal{C} whose objects and arrows are invariant under the action of H .

Definition 3.23. Fix a Burnside functor $F: \mathcal{C} \rightarrow \mathcal{B}_K$ with an external action by G . Let H a subgroup of G . The H -fixed-point functor of F is the functor $F^H: \mathcal{C}^H \rightarrow \mathcal{B}_K$, defined by $F^H(v) = F(v)^H$ (where H acts on $F(v)$ since v is H -fixed, where we have forgotten the decoration of the bijection $\psi_{h,v}$) and where morphisms $F^H(\phi)$ for $\phi \in \mathcal{C}^H$ are sent to $F^H(\phi) = F(\phi)^H$. (Note that H acts on $F(\phi)$ for $\phi \in \text{Hom}(\mathcal{C}^H)$.)

Let us see that F^H is well-defined. Indeed, we need to see that there are canonical associators $\alpha_{\phi_2, \phi_1}^H: F^H(\phi_2) \circ F^H(\phi_1) \rightarrow F^H(\phi_2 \circ \phi_1)$. However, the associator of F restricts to give a bijection of H -fixed-point sets, since the associator commutes with the G -action, by (E-2).

3.3.5 Natural transformations

To relate different functors to the Burnside category, we will need the following notion:

Definition 3.24. A natural transformation $\eta: F_1 \rightarrow F_0$ between 2-functors $F_1, F_0: \mathcal{C} \rightarrow \mathcal{B}_K$ is a strictly unitary 2-functor $\eta: \underline{2} \times \mathcal{C} \rightarrow \mathcal{B}_K$ so that $\eta|_{\{1\} \times \mathcal{C}} = F_1$ and $\eta|_{\{0\} \times \mathcal{C}} = F_0$. A natural transformation of functors $F_1, F_0: \underline{2}^{np} \rightarrow \mathcal{B}_K$ with external action by \mathbb{Z}_p , where \mathbb{Z}_p acts on $\underline{2}^{np}$ by permuting the coordinates, is such an η , itself admitting an external action (where $\underline{2} \times \underline{2}^{np}$ has the product \mathbb{Z}_p -action).

We usually refer to ‘natural transformations with external action’ as ‘natural transformations’ where it will not cause confusion.

For $\mathcal{C} = \underline{2}^n$ or $\underline{2}^{np}$, a natural transformation (functorially) induces a chain map between the chain complexes of Burnside functors, which we write as $\text{Tot}(\eta): \text{Tot}(F_1) \rightarrow \text{Tot}(F_0)$. (In fact, for a natural transformation with external action, $\text{Tot}(\eta)$ is \mathbb{Z}_p -equivariant).

Many of the natural transformations we will encounter will be sub-functor inclusions or quotient functor surjections. Given a functor $F: \underline{2}^{np} \rightarrow \mathcal{B}_K$ with external action,

a *sub-functor with external action* (respectively, *quotient functor*) $H: \underline{2}^{np} \rightarrow \mathcal{B}_K$ is a functor that satisfies:

1. $H(v) \subset F(v)$ for all $v \in \underline{2}^{np}$, and so that the external action of \mathbb{Z}_p restricts to an action on objects.
2. $H(\phi_{u,v}) \subset F(\phi_{u,v})$ for all $u \geq v$, with the source and target maps being restrictions of the corresponding ones on $F(\phi_{u,v})$, and so that the action of \mathbb{Z}_p preserves H (in the natural sense).
3. $s^{-1}(x) \subset H(\phi_{u,v})$ (respectively, $t^{-1}(y) \subset H(\phi_{u,v})$) for all $u \geq v$ and for all $x \in H(u)$ (respectively, $y \in H(v)$).

If H is a sub- (respectively, quotient) functor of F , then there is a natural transformation $H \rightarrow F$ (respectively, $F \rightarrow H$), and the induced chain map $\text{Tot}(H) \rightarrow \text{Tot}(F)$ (respectively, $\text{Tot}(F) \rightarrow \text{Tot}(H)$) is an inclusion (respectively, a quotient map) of chain complexes (in fact, a \mathbb{Z}_p -equivariant map of chain complexes).

Definition 3.25. If J is a sub-functor with external action of $F: \underline{2}^{np} \rightarrow \mathcal{B}_K$, then the functor L defined as $L(v) = F(v) \setminus J(v)$ and $L(\phi_{u,v}) = F(\phi_{u,v}) \setminus J(\phi_{u,v})$ is a quotient functor of F (and vice-versa). Such a sequence

$$J \rightarrow F \rightarrow L$$

is called a *cofibration sequence* of Burnside functors; it induces the short exact sequence

$$0 \rightarrow \text{Tot}(J) \rightarrow \text{Tot}(F) \rightarrow \text{Tot}(L) \rightarrow 0$$

of chain complexes.

3.3.6 Stable equivalence of functors

In the sequel, we will be interested not just in functors $F: \underline{2}^n \rightarrow \mathcal{B}_K$, but in *stable* functors, which are pairs (F, R) for R an element of the real representation ring of G . In case $G = \{1\}$, we view stable functors as pairs (F, r) for r an integer, referring to r copies of the trivial representation. We denote the regular representation of G by $\mathbb{R}(G)$. For an orthogonal G -representation V , write V^+ for its one-point compactification, considered as a pointed space by taking the point at infinity as the basepoint. We will also write $\Sigma^R F$ for (F, R) .

Let $\text{Det}_G = \tilde{H}^*(\mathbb{R}(G)^+)$ as a graded $\mathbb{Z}[G]$ -module. We define the totalization of the stable functor $(F, r\mathbb{R} + s\mathbb{R}(G))$, by $\text{Tot}((F, r\mathbb{R} + s\mathbb{R}(G))) = \text{Tot}(F)[r] \otimes_{\mathbb{Z}} \text{Det}_G^{\otimes s}$, where $\text{Tot}(F)[r]$ denotes the (ordinary) totalization shifted up by r . If $s < 0$, we make sense of the above formula using the (graded) dual of Det_G . In this section we will describe when two such stable functors are equivalent, following Definition 3.6 of [SSS18].

A *face inclusion* ι is a functor $\underline{2}^n \rightarrow \underline{2}^N$ that is injective on objects and preserves the relative gradings. Note that self-equivalences $\iota: \underline{2}^n \rightarrow \underline{2}^n$ are face inclusions. Consider a face inclusion $\iota: \underline{2}^n \rightarrow \underline{2}^N$ and a functor $F: \underline{2}^n \rightarrow \mathcal{B}_K$. The induced functor $F_\iota: \underline{2}^N \rightarrow \mathcal{B}_K$ is uniquely determined by requiring that $F = F_\iota \circ \iota$, and such that for $v \in \underline{2}^N / \iota(\underline{2}^n)$, we have $F_\iota(v) = \emptyset$. For a face inclusion ι , we define $|\iota| = |\iota(v)| - |v|$ for any $v \in \underline{2}^n$, which is independent of v since ι is assumed to preserve relative gradings. For any functor F and face inclusion ι as above,

$$\text{Tot}(F_\iota) \cong \Sigma^{|\iota|} \text{Tot}(F)$$

where the isomorphism is natural up to certain sign choices.

For \mathbb{Z}_p acting on $\underline{2}^{np}, \underline{2}^{Np}$ by cyclic permutation, an *equivariant face inclusion* $\iota: \underline{2}^{np} \rightarrow \underline{2}^{Np}$ will be a face inclusion so that $g\iota = \iota g$ for all $g \in \mathbb{Z}_p$. An equivariant face inclu-

sion induces a \mathbb{Z}_p -equivariant isomorphism between $\text{Tot}(F_\iota)$ and $\text{Det}_{\mathbb{Z}_p}^{|\iota|/p} \otimes \text{Tot}(F)$, also natural up to certain sign choices.

With this background, we state the relevant notion of equivalence for stable functors.

Definition 3.26. Two stable functors $(E_1: \underline{2}^{m_1} \rightarrow \mathcal{B}_K, q_1)$ and $(E_2: \underline{2}^{m_2} \rightarrow \mathcal{B}_K, q_2)$ are *stably equivalent* for $\mathfrak{d}: K \rightarrow \mathbb{Z}_2$ if there is a sequence of stable functors $\{(F_i: \underline{2}^{n_i} \rightarrow \mathcal{B}_K, r_i)\}$ ($0 \leq i \leq \ell$) with $\Sigma^{q_1} E_1 = \Sigma^{r_0} F_0$ and $\Sigma^{q_2} E_2 = \Sigma^{r_\ell} F_\ell$ such that for each pair $\{\Sigma^{r_i} F_i, \Sigma^{r_{i+1}} F_{i+1}\}$, one of the following holds:

1. $(n_i, r_i) = (n_{i+1}, r_{i+1})$ and there is a natural transformation $\eta: F_i \rightarrow F_{i+1}$ or $\eta: F_{i+1} \rightarrow F_i$ such that the induced map $\text{Tot}(\eta)$ is a chain homotopy equivalence.
2. There is a face inclusion $\iota: \underline{2}^{n_i} \hookrightarrow \underline{2}^{n_{i+1}}$ such that $r_{i+1} = r_i - |\iota|$ and $F_{i+1} = (F_i)_\iota$; or a face inclusion $\iota: \underline{2}^{n_{i+1}} \hookrightarrow \underline{2}^{n_i}$ such that $r_i = r_{i+1} - |\iota|$ and $F_i = (F_{i+1})_\iota$.

Two stable functors $(E_1, R_1), (E_2, R_2)$ with $E_i: \underline{2}^{n_i p} \rightarrow \mathcal{B}_K$ with external action by \mathbb{Z}_p (extending the action on $\underline{2}^{n_i p}$ by cyclic permutation) are *externally stably equivalent* if they are related by moves as above, such that the following are satisfied. Moves as in (1) must be natural transformations of functors with external actions, so that $\text{Tot}(\eta^H)$, where η^H is the fixed-point Burnside functor, must be a homotopy equivalence $\text{Tot}(F_i^H) \rightarrow \text{Tot}(F_{i+1}^H)$ or $\text{Tot}(F_{i+1}^H) \rightarrow \text{Tot}(F_i^H)$ for all subgroups $H \subset \mathbb{Z}_p$. Further, the face inclusions in (2) are required to be equivariant. In particular, in the external case for (2), we require $r_{i+1} = r_i - (|\iota|/p)\mathbb{R}(G)$ or $r_i = r_{i+1} - (|\iota|/p)\mathbb{R}(G)$.

We call such a sequence, along with the arrows between $\Sigma^{r_i} F_i$, a *stable equivalence* between the stable functors $\Sigma^{q_1} E_1$ and $\Sigma^{q_2} E_2$. If the sequence is such that the maps η satisfy $\text{Tot}(\eta)$ are chain homotopy equivalences for every choice of degree $\mathfrak{d}: K \rightarrow \mathbb{Z}_2$, we call it a *K-equivariant (stable) equivalence*, and say that $\Sigma^{q_i} E_i$ are *K-equivariantly equivalent*. All external stable equivalences that appear in this paper will be *K-equivariant*.

An external stable equivalence from $\Sigma^{q_1} E_1$ to $\Sigma^{q_2} E_2$ induces a \mathbb{Z}_p -equivariant chain homotopy equivalence $\text{Tot}(\Sigma^{q_1} E_1) \rightarrow \text{Tot}(\Sigma^{q_2} E_2)$, well-defined up to choices of inverses of the chain homotopy equivalences involved in its construction, and an overall sign.

Remark 3.27. We need not restrict to cube categories $\underline{2}^n$ and $\underline{2}^{np}$ in Definition 3.24; this notion makes sense for any small category \mathcal{C} and small category \mathcal{C} with G -action, respectively. However, to define stable equivalences when working with a more general category \mathcal{C} , there is no notion of the totalization. Note that the totalization $\text{Tot}(F)$ is weakly equivalent to the homotopy colimit of $\mathcal{F}F$, viewed in the category of chain complexes, where \mathcal{F} denotes the forgetful functor $\mathcal{B}_K \rightarrow \mathbb{Z}\text{-Mod}$. In particular, the appropriate generalization of Definition 3.26 to more general \mathcal{C} is simply to replace totalizations with homotopy colimits.

We will also need the notion of a *product* of Burnside functors.

Definition 3.28. Given functors $F: \underline{2}^{mp} \rightarrow \mathcal{B}_K$ and $J: \underline{2}^{np} \rightarrow \mathcal{B}$, both with external action by \mathbb{Z}_p , we define the product $F \times J: \underline{2}^{(m+n)p} \rightarrow \mathcal{B}_K$ as follows

1. For $(v_1, v_2) \in \underline{2}^{mp} \times \underline{2}^{np}$, $(F \times J)((v_1, v_2)) = F(v_1) \times J(v_2)$.
2. For all $(u_1, u_2) > (v_1, v_2)$, $(F \times J)(\phi_{(u_1, u_2), (v_1, v_2)}) = F(\phi_{u_1, v_1}) \times J(\phi_{u_2, v_2})$. The decoration on each element of the correspondence is the decoration of $F(\phi_{u_1, v_1})$.
3. For all $(u_1, u_2) > (v_2, v_2) > (w_1, w_2)$, the map $(F \times J)_{(u_1, u_2), (v_1, v_2), (w_1, w_2)}$ is defined:

$$(F \times J)_{(u_1, u_2), (v_1, v_2), (w_1, w_2)}(x_1, x_2) = ((F)_{u_1, v_1, w_1}(x_1), (J)_{u_2, v_2, w_2}(x_2)),$$

where, if $u_i = v_i$ or $v_i = w_i$, we set $(F)_{u_i, v_i, w_i} = \text{Id}$ or $(J)_{u_i, v_i, w_i} = \text{Id}$, respectively.

This defines a strictly unitary lax 2-functor $\underline{2}^{(n+m)p} \rightarrow \mathcal{B}_K$. We have $\text{Tot}(F \times J) = \text{Tot}(F) \otimes \text{Tot}(J)$.

The \mathbb{Z}_p -action on $F \times J$ is given as follows. On objects, $\psi_{g,(v,w)}$ is given by the product action $\psi_{g,(v,w)}: F(v) \times J(w) \rightarrow F(gv) \times J(gw)$, and similarly for the action on correspondences.

3.4 Realizations of Burnside functors

In this section, given a functor $F: \underline{2}^n \rightarrow \mathcal{B}_K$, along with some other choices, we construct an essentially well-defined finite CW spectrum $|F|$, which is an equivariant spectrum in case $K \neq \{1\}$. As a first step, we construct finite CW complexes $\|F\|_V$ for sufficiently large representations V , so that increasing the parameter V corresponds to suspending the CW complex $\|F\|_V$. The finite CW spectrum $|F|$ is then defined from this sequence of spaces. The construction of $\|F\|_V$ depends on some auxiliary choices, but its stable homotopy type does not. Moreover, the spectra constructed from two stably equivalent Burnside functors will be homotopy equivalent. This section, included mostly to set up notation, is almost entirely contained in [SSS18, §4], which itself is mostly a collection of results from [LLS], along with some equivariant topology. The only new material in the present section is Lemma 3.35.

3.4.1 Maps from correspondences

We start with the construction of (ordinary) disk maps, following [LLS, §2.10]¹. Let $B^\ell = \{x \in \mathbb{R}^\ell \mid \|x\| \leq 1\}$, and fix an identification $S^\ell = B^\ell/\partial$, with $\partial := \partial B^\ell$, which we maintain through the sequel, and view S^ℓ as a pointed space. For any subset $B \subset B^\ell$ of the form $B = \{y \in B^\ell \mid \|y - y_0\| \leq c, \|y_0\| + c \leq 1\}$, we note that there is a standard identification of B with a copy of B^ℓ by sending $x \rightarrow c(x - y_0)$ and so we have a standard

¹In previous papers, starting with [LLS], but continuing in [LLS17a],[SSS18], one works with “box maps”. The previous papers could have been executed in very close analogy using disk maps as formulated here, obtaining homotopy-equivalent objects; we prefer disk maps in the present paper as they are more suitable for visualizing the group action.

identification $S^\ell = B/\partial B$. In the sequel, by a subdisk $B \subset B^\ell$ we will mean a subset B as above, along with the standard identification $\phi: B^\ell \rightarrow B$ (that is, we think of a subdisk as a subset of B^ℓ , along with the *standard framing*).

Given a collection (indexed by $\{1, \dots, t\}$) of sub-disks $B_1, \dots, B_t \subset B$ with disjoint interiors, there is an induced map

$$S^\ell = B/\partial B \rightarrow B/(B \setminus (\mathring{B}_1 \cup \dots \cup \mathring{B}_t)) = \bigvee_{a=1}^t B_a/\partial B_a = \bigvee_{a=1}^t S^\ell \rightarrow S^\ell. \quad (3.29)$$

The last map is the identity on each summand, so that the composition has degree t . As observed in [LLS], this construction is continuous in the position of the sub-disks. We let $E(B, t)$ denote the space of (indexed) disks with disjoint interiors in B , and have a continuous map $E(B, t) \rightarrow \text{Map}(S^\ell, S^\ell)$.

We can generalize the above procedure to associate a map of spheres to a map of sets $A \rightarrow Y$, as follows. Say we have chosen sub-disks $B_a \subset B$ with disjoint interiors, for $a \in A$. Then we have a map:

$$S^\ell = B/\partial B \rightarrow B/(B \setminus (\bigcup_{a \in A} \mathring{B}_a)) = \bigvee_{a \in A} B_a/\partial B_a = \bigvee_{a \in A} S^\ell \rightarrow \bigvee_{y \in Y} S^\ell \quad (3.30)$$

where the last map is built using the map of sets $A \rightarrow Y$.

More generally, we can also create maps from a correspondence of sets, as follows. Fix a correspondence A from X to Y with source map s and target map t . Say that we also have a collection of disks B_x for $x \in X$. Finally, we also choose a collection of sub-disks $B_a \subset B_{s(a)}$ with disjoint interiors, for $a \in A$. We then have an induced map

$$\bigvee_{x \in X} S^\ell \rightarrow \bigvee_{y \in Y} S^\ell, \quad (3.31)$$

by applying, on B_x , the map associated to the set map $s^{-1}(x) \rightarrow Y$. A map as in

Equation (3.31) is said to *refine* the correspondence A . Let $E(\{B_x\}, s)$ be the space of collections of labeled sub-disks $\{B_a \subset B_{s(a)} \mid a \in A\}$ with disjoint interiors. Then, choosing a correspondence (A, s, t) (so that A and s are those appearing in the definition of $E(\{B_x\}, s)$ —note that the definition of $E(\{B_x\}, s)$ does not involve the target map t)—Equation (3.31) gives a map $E(\{B_x\}, s) \rightarrow \text{Map}(\bigvee_{x \in X} S^\ell, \bigvee_{y \in Y} S^\ell)$. We write

$$\Phi(e, A) \in \text{Map}\left(\bigvee_{x \in X} S^\ell \rightarrow \bigvee_{y \in Y} S^\ell\right) \quad (3.32)$$

for the map associated to $e \in E(\{B_x\}, s)$ and a compatible correspondence (A, s, t) . One of the main points is that, for any disk map $\Phi(e, A)$ refining A , the induced map on the ℓ^{th} homology agrees with the abelianization map

$$\mathcal{A}(A): \mathcal{A}(X) = \tilde{H}_\ell(\bigvee_{x \in X} S^\ell) \rightarrow \mathcal{A}(Y) = \tilde{H}_\ell(\bigvee_{y \in Y} S^\ell).$$

We now indicate a further generalization of disk maps to cover decorated correspondences, along with a choice of representation $\mathfrak{r}: K \rightarrow \text{Homeo}(B^\ell)$, so that the topological degree of \mathfrak{r} is $\mathfrak{d}: K \rightarrow \mathbb{Z}_2$. Fix a decorated correspondence (A, s, t, σ) from X to Y , and let B_x , $x \in X$ be some collection of disks. Fix a collection of K -labeled subdisks $\phi_a: B_a \hookrightarrow B_{s(a)}$ for $a \in A$. There is an induced map just as in Equation (3.31), but whose construction depends on the decoration σ , as follows. For $x \in X$, we have a set map $s^{-1}(x) \rightarrow Y$, along with decorations for each element of $s^{-1}(x)$. We modify the disk map refining $s^{-1}(x) \rightarrow Y$ (without decoration) by precomposing with $\mathfrak{r}(\sigma(a))$:

$$S^\ell = B/\partial B \rightarrow B/(B \setminus (\bigcup_{a \in A} \mathring{B}_a)) = \bigvee_{a \in A} B_a/\partial B_a \xrightarrow{\bigvee \mathfrak{r}(\sigma(a))} \bigvee_{a \in A} B_a/\partial B_a = \bigvee_{a \in A} S^\ell \rightarrow \bigvee_{y \in Y} S^\ell.$$

We say that a map constructed this way \mathfrak{r} -*refines* (or, when \mathfrak{r} is clear from context, simply *refines*) the decorated correspondence (A, s, t, σ) . As before, we can regard the

above construction as a map

$$\Phi(e, A) \in \text{Map}\left(\bigvee_{x \in X} S^\ell, \bigvee_{y \in Y} S^\ell\right),$$

where $e \in E(\{B_x\}, s)$, and (A, s, t, σ) is a compatible decorated correspondence. Once again, the induced map on the ℓ^{th} homology agrees with the \mathfrak{d} -abelianization map.

We will assume henceforth that $B = B(V)$ is the unit disk of some orthogonal representation V of K .

Let $E_{K,V}(\{B_x\}, s)$ denote the set of disk embeddings in $E(\{B_x\}, s)$ whose centers lie in $B(V)^K$.

Lemma 3.33. *[cf. Lemma 4.5 [SSS18]] Consider $s: A \rightarrow X$. If $\dim(V^K) \geq k$ then $E_{K,V}(\{B_x\}, s)$ is $(k-2)$ -connected.*

Proof. The proof is analogous to [LLS, Lemma 2.29] or [SSS18, Lemma 4.5]. \square

Lemma 3.34. *[cf. Lemma 4.6 [SSS18]] Fix an orthogonal K -representation \mathfrak{r} . If $e \in E(\{B_x\}, s_A)$ is compatible with a decorated correspondence A from X to Y , and $f \in E(\{B_y\}, s_B)$ is compatible with a decorated correspondence B from Y to Z , then there is a unique $f \circ_{\mathfrak{r}} e \in E(\{B_x\}, s_{B \circ A})$ compatible with $B \circ A$, so that $\Phi(f \circ_{\mathfrak{r}} e, B \circ A) = \Phi(f, B) \circ \Phi(e, A)$. Moreover, this assignment $E(\{B_y\}, s_B) \times E(\{B_x\}, s_A) \rightarrow E(\{B_x\}, s_{B \circ A})$ is continuous and sends $E_{K,V}(\{B_y\}, s_B) \times E_{K,V}(\{B_x\}, s_A)$ to $E_{K,V}(\{B_x\}, s_{B \circ A})$.*

Proof. For $(b, a) \in B \times_Y A$ with decorations g, h respectively, define $B_{(b,a)}$ to be the sub-disk whose image is

$$B_b \hookrightarrow B_{s_B(b)=t_A(a)} \xrightarrow{\mathfrak{r}(h^{-1})} B_{s_B(b)} = B_a \hookrightarrow B_{s_A(a)}.$$

Note that the embedding of the disk is not that given by the above composition (rather, we take the *standard framed* disk with the same image). This defines $f \circ_{\mathfrak{r}} e$ as

the image of (f, e) under the assignment $E(\{B_y\}, s_B) \times E(\{B_x\}, s_A) \rightarrow E(\{B_x\}, s_{B \circ A})$. It follows from the definitions that $\Phi(f \circ_{\mathbf{r}} e, B \circ A) = \Phi(f, B) \circ \Phi(e, A)$.

Finally, consider the restriction of the assignment to $E_{K,V}(\{B_y\}, s_B) \times E_{K,V}(\{B_x\}, s_A)$. It is clear that the above construction takes disks centered on V^K to disks centered on V^K , completing the proof. □

Note that if K is abelian, then for $e \in E_{K,V}(\{B_x\}, s)$, the induced map $\Phi(e, A)$ is K -equivariant.

Fix a finite group G and let \mathcal{U} be a complete $K \times G$ -universe [May96]. The set of finite-dimensional subspaces of \mathcal{U} is partially ordered by inclusion. For a finite-dimensional subspace V of \mathcal{U} , let $B(V)$ denote the unit ball.

Lemma 3.35. *Let H a subgroup of G as above and A a correspondence from X to Y , possibly K -decorated. Say that H acts on X and Y , compatibly with source and target maps, by K -decorated bijections ψ_h for $h \in H$, and we are given the data of 2-morphisms $\psi_{h,A}: \psi_h \circ A \rightarrow A \circ \psi_h$ for all $h \in H$.*

Further, define an action of H on $E(\{B_x\}, s)$ by sending a collection of embedded disks $\{\phi_a: B_a \rightarrow B_{s(a)}\}_{a \in A}$ to $\{g\phi_a g^{-1}: B_{ga} \rightarrow B_{gs(a)}\}_{a \in A}$. Here g takes $B_a \rightarrow B_{ga}$, by the action of $g \in G$ on $B(V)$ using the identifications $B_{ga} = B(V) = B_a$. That is, say the correspondence $\psi_g: Y \rightarrow Y$ takes $y \in Y$ to $gy \in Y$. Then, g acts by $B_a = B(V) \xrightarrow{g} B(V) = B_{ga}$, and similarly for g^{-1} (using the decorations of the action on X). For any $N > 0$, for all sufficiently large finite-dimensional subspaces V of \mathcal{U} , $E_{K,V}(\{B_x(V)\}, s)^H$ is N -connected (and, in particular, nonempty).

Proof. We must first see that the formula for the H -action indeed gives a well-defined action. By definition, it takes disks to disks, but it is also necessary that it take *framed* disks to framed disks; this is clear from the definitions. (Note, however, that for an arbitrary H -action on a ball B , this condition need not be satisfied).

We write out the argument for $K = \{1\}$; for more general K one need only replace V with V^K throughout. We also reduce to the case $X = \{x\}$ as follows. First, label $A = \{a_1^1, \dots, a_{k_1}^1, \dots, a_1^n, \dots, a_{k_n}^n\}$, where $s(a_{j_1}^{i_1}) = s(a_{j_2}^{i_2})$ if and only if $i_1 = i_2$. Observe that to an element of $E(\{B_x(V)\}, s)$, we may associate a tuple $(z_1^1, \dots, z_{k_n}^n) \in V^{\times(\sum k_i)}$ by taking centers, and $E(\{B_x(V)\}, s)$ is equivariantly homotopy-equivalent to $V^{\times \sum k_i} - \Delta$ where Δ is the set of tuples (z_i) for which there is some pair $i \neq j$ with $z_i = z_j$. Let Π denote the projection map $E(\{B_x(V)\}, s) \rightarrow V^{\times \sum k_i} - \Delta$. Now, if for some $v \in V^{\times \sum k_i}$ we have $v_{\ell_1}^i = v_{\ell_2}^i$ with $\ell_1 \neq \ell_2$, then $gv_{\ell_1}^i = gv_{\ell_2}^i$ by definition. Thus, a H -fixed tuple $(z) \in V^{\times \sum k_i}$ (where H acts on $V^{\times \sum k_i}$ as in the statement of the lemma) is in $\text{Im} \Pi|_{E(\{B_x(V)\}, s)^H}$ if and only if, for x running over any set of representatives for the orbits of X under H , there are no $a_1 \neq a_2 \in A$ so that $s(a_1) = s(a_2) = x$ and $z_1 = z_2$ (where $z_i \in V$ are the centers of the disks corresponding to a_i). That is, we may assume $X = \{x\}$, by possibly replacing H with a subgroup $H' \subset H$.

Fix $X = \{x\}$ and let Ω denote the set of orbits of the G -action on A , and choose some identification $\Omega \cong \{1, \dots, n\}$. Choose a collection of representatives $\{a_i\}_{1 \leq i \leq n}$ for the orbits. For each a_i , let $S_i \subset H$ be the stabilizer. Define $\langle g \rangle \subset G$ to be the subgroup of G generated by $g \in G$. Let $B = B(V)$ for some V sufficiently large. Let $E'(\{B_x\}, s)$ be the set of tuples $(x_1, \dots, x_n) \in V^{\times n}$ so that $x_i \in V^{S_i}$ and so that, for all $g \in G$ which are not in S_i , $x_i \notin V^{\langle g \rangle}$, and so that $(x_1, \dots, x_n) \notin \Delta$ where Δ is the set of tuples (x_i) for which there is some pair $i \neq j$ with $g_1 x_i = g_2 x_j$ for some g_1, g_2 . We observe that associated to any element of $E(\{B_x\}, s)^H$ we obtain a tuple of points in $B(V)$ by taking centers of the embedded disks; in fact, this gives a map $E(\{B_x\}, s)^H \rightarrow E'(\{B_x\}, s)$ with contractible fibers. The condition that $x_i \notin V^{\langle g \rangle}$ is necessary, since an element $e \in E(\{B_x\}, s)$ must consist of disks whose interiors are disjoint; indeed, if $x_i \in V^{\langle g \rangle}$ for $g \notin S_i$, then the interiors of the disks B_{a_i} and B_{ga_i} would intersect nontrivially.

That is, $E'(\{B_x\}, s) = \prod_{i=1}^n V^{S_i} / D$ where

$$D = \Delta \cup \bigcup_{i=1}^n \left((V^{S_1} \times V^{S_2} \times \dots \times V^{S_{i-1}}) \times \bigcup_{g \notin S_i} V^{\langle g \rangle} \times (V^{S_{i+1}} \times \dots \times V^{S_n}) \right).$$

For given $N > 0$, to show that $E'(\{B_x\}, s)$ (and therefore also $E(\{B_x\}, s)^H$) is N -connected, it suffices to show that D has arbitrarily high codimension in $\prod_{i=1}^n V^{S_i}$. This can be achieved by constructing a suitably large V in \mathcal{U} .

As \mathcal{U} is complete, it contains infinitely many copies of the regular representation $\mathbb{R}[G]$ of our finite group G . Recall that $\mathbb{R}[G]$ satisfies the following two properties:

1. It contains a copy of the trivial representation; every $g \in G$ acts trivially on this 1-dimensional component.
2. For any $1 \neq g \in G$, g acts nontrivially on some irreducible component of $\mathbb{R}[G]$.

Given $1 \neq g \in G$, these two facts show that both the dimension and the codimension of $\mathbb{R}[G]^{\langle g \rangle}$ are at least 1.

As D is the union of finitely many pieces, it suffices to show that each piece has arbitrarily high codimension, say at least $N + 2$. Choose $V \cong \mathbb{R}[G]^{\oplus N+2}$, so that $V^{\langle g \rangle}$ has dimension and codimension at least $N + 2$. It is now clear that the non- Δ pieces of D have codimension at least $N + 2$. To see that Δ also has high codimension, observe that Δ is the (finite) union of subsets homoeomorphic to diagonals $\Delta_{i,j} = \{(x, x) \in V^{S_i} \times V^{S_j}\}$ for $i \neq j$, thickened by the remaining components $\prod_{k \neq i,j} V^{S_k}$. The dimension of $\Delta_{i,j}$ is at most $\min(\dim V^{S_i}, \dim V^{S_j})$, so its codimension is at least $\max(\dim V^{S_i}, \dim V^{S_j})$, which is at least $N + 2$. The codimension of $\Delta_{i,j} \subset V^{S_i} \times V^{S_j}$ is the same as the codimension of $\Delta_{i,j} \times (\prod_{k \neq i,j} V^{S_k}) \subset \prod_{k=1}^n V^{S_k}$.

The same argument applies to any representation containing V , and so the result now follows. □

Lemma 3.36. *Maintain the notation from Lemma 3.35. For $e \in E_{K,V}(\{B_x(V)\}, s)^H$, the induced map $\Phi(e, A)$ is $K \times H$ -equivariant.*

Proof. This follows from the definition of disk maps, as well as the definition of the H -action on $E(\{B_x(V)\}, s)$ in Lemma 3.35. \square

3.4.2 Equivariant topology

Let Top_* be the category of well-based topological spaces; we will usually work with finite CW complexes. A *weak equivalence* $X \rightarrow Y$ is a map that induces isomorphisms on all homotopy groups; typically our spaces are all simply connected, when the definition reduces to being isomorphisms on all homology groups. Homotopy equivalence is a special case of weak equivalence, and for CW complexes (the case at hand), the two notions are equivalent.

We will sometimes also work with spaces equipped with an action by a fixed finite group G , and all maps are G -equivariant, forming a category $G\text{-Top}_*$. We also require that the inclusions of fixed points $X^H \rightarrow X^{H'}$, for all subgroups $H' < H$ of G , are cofibrations; in our case, all the spaces will carry CW structures so that the actions are CW actions—that is, each group element simply permutes the cells and respects the attaching maps. A map $X \rightarrow Y$ is called a *weak equivalence* if the induced map $X^H \rightarrow Y^H$ is a weak equivalence for all subgroups H of G . A homotopy of G -maps $X \rightarrow Y$ is an extension to a G -equivariant map $X \times I \rightarrow Y$, where $X \times I$ is given a G -structure by $g(x, i) = (gx, i)$, and we have the usual notion of a nullhomotopy. A homotopy equivalence in $G\text{-Top}_*$ induces a weak equivalence. For G -CW complexes (the case at hand), the two notions are equivalent by the G -Whitehead theorem, see [GM95, Theorem 2.4]. For G -CW complexes, a weak equivalence $X \rightarrow Y$ induces a weak equivalence between quotients of fixed points, $X^{H'}/X^H \rightarrow Y^{H'}/Y^H$, for all subgroups $H' < H$ of G , and between orbit spaces, $X/H \rightarrow Y/H$, for all subgroups H of G .

We will also need the concept of a G -universe \mathcal{U} , that is, a countably infinite-dimensional real inner product space with an action of G by linear isometries [May96]. The space \mathcal{U} is called *complete* if it contains an infinite number of copies of all finite-dimensional irreducible representations of G .

3.4.3 Homotopy coherence

In this section, we briefly review homotopy colimits and homotopy coherent diagrams following [LLS, §2.9].

We recall the notion of a homotopy coherent diagram, which is the data from which a homotopy colimit is constructed. A homotopy coherent diagram is intuitively a diagram $F: \mathcal{C} \rightarrow K\text{-Top}_*$ which is not commutative, but commutative up to homotopy, and the homotopies themselves commute up to higher homotopy, and so on, and for which all the homotopies and higher homotopies are viewed as part of the data of the diagram. Precisely, we have the following.

Definition 3.37 ([Vog73, Definition 2.3]). A homotopy coherent diagram $F: \mathcal{C} \rightarrow K\text{-Top}_*$ assigns to each $x \in \mathcal{C}$ a space $F(x) \in K\text{-Top}_*$, and for each $n \geq 1$ and each sequence

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$$

of composable morphisms in \mathcal{C} a continuous map

$$F(f_n, \dots, f_1): [0, 1]^{n-1} \times F(x_0) \rightarrow F(x_n)$$

with $F(f_n, \dots, f_1)([0, 1]^{n-1} \times \{*\}) = *$. These maps are required to satisfy the following

compatibility conditions:

$$F(f_n, \dots, f_1)(t_1, \dots, t_{n-1}) = \begin{cases} F(f_n, \dots, f_2)(t_2, \dots, t_{n-1}), & f_1 = \text{Id} \\ F(f_n, \dots, \hat{f}_i, \dots, f_1)(t_1, \dots, t_{i-1} \cdot t_i, \dots, t_{n-1}), & f_i = \text{Id}, 1 < i < n \\ F(f_{n-1}, \dots, f_1)(t_1, \dots, t_{n-2}), & f_n = \text{Id} \\ F(f_n, \dots, f_{i+1})(t_{i+1}, \dots, t_{n-1}) \circ F(f_i, \dots, f_1)(t_1, \dots, t_{i-1}), & t_i = 0 \\ F(f_n, \dots, f_{i+1} \circ f_i, \dots, f_1)(t_1, \dots, \hat{t}_i, \dots, t_{n-1}), & t_i = 1. \end{cases} \quad (3.38)$$

When \mathcal{C} does not contain any non-identity isomorphisms, homotopy coherent diagrams may be defined only in terms of non-identity morphisms and the last two compatibility conditions.

Given a homotopy coherent diagram, we can define its *homotopy colimit* in $K\text{-Top}_*$, quite concretely, as follows:

Definition 3.39 ([Vog73, §5.10]). Given a homotopy coherent diagram $F: \mathcal{C} \rightarrow K\text{-Top}_*$ the *homotopy colimit* of F is defined by

$$\text{hocolim } F = \{*\} \amalg \coprod_{n \geq 0} \coprod_{x_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} x_n} [0, 1]^n \times F(x_0) / \sim, \quad (3.40)$$

where the equivalence relation \sim is given as follows:

$$(f_n, \dots, f_1; t_1, \dots, t_n; p) \sim \begin{cases} (f_n, \dots, f_2; t_2, \dots, t_n; p), & f_1 = \text{Id} \\ (f_n, \dots, \hat{f}_i, \dots, f_1; t_1, \dots, t_{i-1} \cdot t_i, \dots, t_n; p), & f_i = \text{Id}, i > 1 \\ (f_n, \dots, f_{i+1}; t_{i+1}, \dots, t_n; F(f_i, \dots, f_1)(t_1, \dots, t_{i-1}, p)), & t_i = 0 \\ (f_n, \dots, f_{i+1} \circ f_i, \dots, f_1; t_1, \dots, \hat{t}_i, \dots, t_n; p), & t_i = 1, i < n \\ (f_{n-1}, \dots, f_1; t_1, \dots, t_{n-1}; p), & t_n = 1 \\ *, & p = *. \end{cases}$$

When \mathcal{C} does not contain any non-identity isomorphisms, homotopy colimits may be defined only in terms of non-identity morphisms and the last four equivalence relations.

That is,

$$\text{hocolim } F = \{*\} \amalg \coprod_{n \geq 0} \coprod_{\substack{x_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} x_n \\ \forall i \in \{1, \dots, n\}, f_i \neq \text{Id}}} [0, 1]^n \times F(x_0) / \sim',$$

where \sim' , in the case \mathcal{C} has no non-identity isomorphisms, is the last four cases of the definition of \sim .

In this paper, the categories \mathcal{C} will have no non-identity isomorphisms, so we will work with the latter formulation.

We will occasionally need:

Definition 3.41. [Definition 2.6, [Vog73]] A *homomorphism* of homotopy coherent diagrams $F_1, F_0: \mathcal{C} \rightarrow K\text{-Top}_*$ is a collection of maps $\phi_x: F_1(x) \rightarrow F_0(x)$ for each $x \in \text{Ob}(\mathcal{C})$, so that

$$F_0(f_n, \dots, f_1)(t_1, \dots, t_{n-1}) \circ \phi_x = \phi_y \circ F_1(f_n, \dots, f_1)(t_1, \dots, t_{n-1}),$$

where $f_n \circ \dots \circ f_1: x \rightarrow y \in \mathcal{C}$, for all t_i .

A homotopy-coherent diagram may itself be viewed as a commutative diagram from an auxiliary category as in [Vog73, Definition 2.3], and a homomorphism of homotopy coherent diagrams is a homomorphism (of diagrams, in the usual sense) of the associated commutative diagrams from the auxiliary category.

We will need the following properties:

(ho-1) Suppose that $F_0, F_1: \mathcal{C} \rightarrow K\text{-Top}_*$ are homotopy coherent diagrams and $\eta: F_1 \rightarrow F_0$ is a natural transformation, that is, a homotopy coherent diagram

$$\eta: \underline{2} \times \mathcal{C} \rightarrow K\text{-Top}_*$$

with $\eta|_{\{i\} \times \mathcal{C}} = F_i$, $i = 0, 1$. Then η induces a map $\text{hocolim } \eta: \text{hocolim } F_1 \rightarrow \text{hocolim } F_0$, well-defined up to homotopy, according to [Vog73, Theorem 5.12]. If $\eta(x)$ is a weak equivalence for each $x \in \mathcal{C}$ —we will call such an η a weak equivalence $F_1 \rightarrow F_0$ —then $\text{hocolim } \eta$ is a weak equivalence as well.

When the spaces involved are K -CW complexes (the case at hand), a weak equivalence $\eta: F_1 \rightarrow F_0$ is also a homotopy equivalence [Vog73, Proposition 4.6], that is, there exists $\zeta, \zeta': F_0 \rightarrow F_1$ and

$$\mathfrak{h}, \mathfrak{h}': \{2 \rightarrow 1 \rightarrow 0\} \times \mathcal{C} \rightarrow K\text{-Top}_*,$$

with $\mathfrak{h}|_{\{2 \rightarrow 1\} \times \mathcal{C}} = \eta$, $\mathfrak{h}|_{\{1 \rightarrow 0\} \times \mathcal{C}} = \zeta$, $\mathfrak{h}|_{\{2 \rightarrow 0\} \times \mathcal{C}} = \text{Id}_{F_0}$, and $\mathfrak{h}'|_{\{2 \rightarrow 1\} \times \mathcal{C}} = \zeta'$, $\mathfrak{h}'|_{\{1 \rightarrow 0\} \times \mathcal{C}} = \eta$, $\mathfrak{h}'|_{\{2 \rightarrow 0\} \times \mathcal{C}} = \text{Id}_{F_1}$.

(ho-2) A homomorphism $F_1 \rightarrow F_0: \mathcal{C} \rightarrow K\text{-Top}_*$ of homotopy coherent diagrams induces a K -equivariant map $\text{hocolim } F_1 \rightarrow \text{hocolim } F_0$, compatible with (ho-1), as in [Vog73, Proposition 7.1].

(ho-3) For any subgroup H of K , define the fixed-point diagram $F^H: \mathcal{C} \rightarrow \text{Top}_*$ by

setting $F^H(x)$ to be the fixed points $F(x)^H$. Then there is a natural homeomorphism

$$\mathrm{hocolim}(F)^H \simeq \mathrm{hocolim}(F^H).$$

(ho-4) Suppose that $F: \mathcal{C} \rightarrow \mathrm{Top}_*$ and $G: \mathcal{D} \rightarrow \mathrm{Top}_*$. Then there is an induced functor $F \wedge G: \mathcal{C} \times \mathcal{D} \rightarrow \mathrm{Top}_*$ with $(F \wedge G)(v \times w) = F(v) \wedge G(w)$. Then there is a natural (in homomorphisms of homotopy coherent diagrams) weak equivalence $(\mathrm{hocolim} F) \wedge (\mathrm{hocolim} G) \rightarrow \mathrm{hocolim}(F \wedge G)$.

(ho-5) Let $L: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between small categories. Given $d \in \mathrm{Ob}(\mathcal{D})$, the *undercategory* of d is as follows. It has objects $\{(c, f) \mid c \in \mathcal{C}, f: d \rightarrow L(c)\}$, and arrows $\mathrm{Hom}((c, f), (c', f')) = \{g: c \rightarrow c' \mid f' = L(g) \circ f\}$. We write $d \downarrow L$ for the undercategory of d . The functor L is called *homotopy cofinal* if for each $d \in \mathrm{Ob}(\mathcal{D})$, the undercategory $d \downarrow L$ has contractible nerve.

For a homotopy coherent diagram $F: \mathcal{D} \rightarrow \mathrm{Top}_*$, there is an induced homotopy coherent diagram $F \circ L: \mathcal{C} \rightarrow \mathrm{Top}_*$. Require that $F(j)$ is cofibrant for all $j \in \mathrm{Ob}(\mathcal{D})$. If L is homotopy cofinal, then the natural map

$$\mathrm{hocolim} F \circ L \rightarrow \mathrm{hocolim} F$$

is a homotopy equivalence. This follows from the version for homotopy limits in [BK72]; cf. [LLS, (ho-4), §2.9].

3.4.4 Little disks refinement

With this background, we are ready to review the little box realization construction of [LLS, §5] and generalize to functors to \mathcal{B}_K . Assume from now on that K is abelian.

Definition 3.42. Fix a small category \mathcal{C} and a strictly unitary 2-functor $F: \mathcal{C} \rightarrow \mathcal{B}_K$. A *spatial refinement* of F modeled on V , for V an orthogonal K -representation, is a homotopy coherent diagram $\tilde{F}: \mathcal{C} \rightarrow K\text{-Top}_*$ such that

1. For any $u \in \mathcal{C}$, $\tilde{F}(u) = \bigvee_{x \in F(u)} B(V)/\partial B(V)$.
2. For any sequence of morphisms $u_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} u_n$ in \mathcal{C} and any $(t_1, \dots, t_{n-1}) \in [0, 1]^{n-1}$ the map

$$\tilde{F}_k(f_n, \dots, f_1)(t_1, \dots, t_{n-1}): \bigvee_{x \in F(u_0)} B(V)/\partial B(V) \rightarrow \bigvee_{x \in F(u_n)} B(V)/\partial B(V)$$

is a K -equivariant disk map refining the correspondence $F(f_n \circ \cdots \circ f_1)$, which is naturally isomorphic to $F(f_n) \times_{F(u_{n-1})} \cdots \times_{F(u_1)} F(f_1)$.

This definition extends [LLS, Definition 5.1] and [SSS18, Definition 4.11].

The main technical result that makes it possible to construct spatial refinements from Burnside functors is as follows.

Proposition 3.43. [cf Proposition 4.9 [SSS18], Proposition 5.2 [LLS]] *Let \mathcal{C} be a small category in which every sequence of composable non-identity morphisms has length at most n , and let $F: \mathcal{C} \rightarrow \mathcal{B}_K$ be a strictly unitary 2-functor.*

1. *For V sufficiently large, there is a spatial refinement of F modeled on V .*
2. *For V sufficiently large, any two spatial refinements of F modeled on V are weakly equivalent.*
3. *If \tilde{F} is a spatial refinement of F modeled on V then the result of suspending each $\tilde{F}(u)$ and $\tilde{F}(f_n, \dots, f_1)$ by W gives a spatial refinement of F modeled on $V \oplus W$.*

Proof. This is entirely analogous to the proof of Proposition 4.9 of [SSS18] □

3.4.5 Realization of cube-shaped diagrams

Finally in this section we will discuss how to construct a CW complex $\|F\|$, and then a CW spectrum $|F|$, from a given diagram $F: \underline{2}^n \rightarrow \mathcal{B}_K$. We assume in this section that K is abelian. Let $\underline{2}_+$ be the category with objects $\{0, 1, *\}$ and unique non-identity morphisms $1 \rightarrow 0$ and $1 \rightarrow *$, and let $\underline{2}_+^n = (\underline{2}_+)^n \amalg *$ where, for $v \in \underline{2}_+^n - \{0^n\}$, there is a unique arrow $v \rightarrow *$, and $\text{Hom}(0^n, *) = \emptyset$.

Let $\tilde{F}: \underline{2}_+^n \rightarrow K\text{-Top}_*$ be the spatial refinement of F modeled on a K -representation V , and let $\tilde{F}^+: \underline{2}_+^n \rightarrow \text{Top}_*$ be the diagram obtained from \tilde{F} by setting $\tilde{F}^+(*) = \text{pt.}$ Let $\|F\|_V$ be the homotopy colimit of \tilde{F}^+ (we will usually suppress V from the notation). Sometimes we write $\|\tilde{F}^+\|$ to indicate dependence on the choice of spatial refinement. We call $\|F\|$ the *realization* of $F: \underline{2}^n \rightarrow \mathcal{B}_K$.

Corollary 3.44. *[cf. Corollary 5.6 [LLS] and Corollary 4.14 [SSS18]] For V sufficiently large, the realization $\|F\|_V$ is well-defined up to weak equivalence in $K\text{-Top}_*$.*

Proof. This follows from Proposition 3.43 and properties of homotopy colimits (ho-1). □

The homotopy colimit $\|F\|$ may be given several CW structures. First, from Definition 3.39, there is the *standard* CW structure, with cells $[0, 1]^m \times B_x$, parameterized by tuples (f_m, \dots, f_1) subject to some relations. Usually, this will not even be a K -CW decomposition (as some cells may be, for example, fixed by the action of K , but not fixed pointwise, as in the definition of a K -CW structure).

We have a second CW structure on $\|F\|$, the *fine* structure, which is obtained from the standard structure by subdividing each cell $[0, 1]^m \times B_x$ into K -CW cells.

There is also the *coarse* cell structure of [LLS, Section 6]. There they construct a CW structure on $\|F\|$ for F an (unsigned) Burnside functor, with cells formed by taking unions of standard cells, so that there is exactly one (non-basepoint) cell $\mathcal{C}(x)$ for each

$x \in \Pi_u F(u)$. In more detail, if F_x denotes the Burnside sub-functor of F generated by x , then the subcomplex $\|F_x\|$ of $\|F\|$ is the image of the cell $\mathcal{C}(x)$. The coarse cell structure generalizes in a straightforward way to K -equivariant realizations; but it is not an equivariant CW-structure.

Proposition 3.45. *If $F: \underline{2}^n \rightarrow \mathcal{B}_K$ and $\mathfrak{d}: K \rightarrow \mathbb{Z}_2$, then the shifted reduced cellular complex $\tilde{C}_{\text{cell}}(\|F\|_V)[- \dim V]$ is isomorphic to the totalization $\text{Tot}_{\mathfrak{d}}(F)$ with the cells mapping to the corresponding generators. If $\eta: F_1 \rightarrow F_0$ is a natural transformation of Burnside functors, then the map $\|F_1\| \rightarrow \|F_0\|$ is cellular, and the induced cellular chain map agrees with $\text{Tot}(\eta)$.*

Proof. This follows as Proposition 4.16 of [SSS18]. \square

We package the output of this construction as a *finite CW spectrum*, by which we mean a pair (X, W) , for X a pointed K -CW complex and W a formal linear combination of elements of some (fixed) complete K -universe \mathcal{U} . Such a pair can be viewed as an object of the Spanier-Whitehead category, or as $\Sigma^W(\Sigma^\infty X)$, the W -suspension of the suspension spectrum of the K -CW complex X . We define the (*spectrum*) *realization* of a stable Burnside functor $(F: \underline{2}^n \rightarrow \mathcal{B}_K, W)$ as the finite CW spectrum $|\Sigma^W F| = (\|F\|_V, W - V)$.

We record a result of [SSS18] (there it is proved for $K = \mathbb{Z}_2$; the more general proof is no different):

Proposition 3.46. *[Lemma 4.17 [SSS18]] Let $(F: \underline{2}^n \rightarrow \mathcal{B}_K, W)$ be a stable Burnside functor. The spectrum realization $|\Sigma^W F|$ is well-defined up to K -equivariant stable homotopy equivalence. For stable Burnside functors (F_i, W_i) for $i = 1, 2$, a K -equivariant stable equivalence $\Sigma^{W_1} F_1 \rightarrow \Sigma^{W_2} F_2$ induces a K -equivariant homotopy equivalence*

$$|\Sigma^{W_1} F_1| \rightarrow |\Sigma^{W_2} F_2|$$

well-defined up to K -equivariant homotopy equivalence.

3.5 External actions and realization

Our goal in the following will be to show that, for a Burnside functor F with an external action ψ , a suitable realization of F admits a G -action, and the fixed-point set can be explicitly described as a realization of yet another Burnside functor. In Section 3.5.1 we deal with some generalities on homotopy coherent diagrams, then specialize to homotopy coherent diagrams from Burnside functors in Section 3.5.2. Throughout this section we assume that K is an abelian group.

3.5.1 External actions on homotopy coherent diagrams

Definition 3.47. Let $F: \mathcal{C} \rightarrow \text{Top}_*$ a homotopy coherent diagram, where \mathcal{C} is a small category so that there is some n for which each sequence of composable non-identity morphisms has length at most n . Say that a finite group G acts on \mathcal{C} . An *external action* $\bar{\psi}$ of G on F is defined as follows. An external action consists of a map $\bar{\psi}: G \rightarrow \text{Homeo}(\bigvee_{c \in \text{Ob}(\mathcal{C})} F(c))$ lifting the group action ψ of G on $\text{Ob}(\mathcal{C})$ (and preserving the basepoint). The action $\bar{\psi}$ is required to ‘commute with composition’ in the following sense:

$$\bar{\psi}_g(F(f_i, \dots, f_1)(t_1, \dots, t_{i-1})(y)) = F(\psi_g(f_i), \dots, \psi_g(f_1))(t_1, \dots, t_{i-1})(\bar{\psi}_g y), \quad (3.48)$$

for all $g \in G$ and $y \in F(c)$. For a functor $F: \mathcal{C} \rightarrow K\text{-Top}_*$, an external action on F is as above but further requiring that the K and G actions commute.

Remark 3.49. A homotopy coherent diagram with external action by G may be thought of as an analogue of a G -space in the category of homotopy coherent diagrams. First,

note that a pointed G -space X may be viewed as a functor $\underline{X}: BG \rightarrow \text{Top}_*$, where BG is the category with one object, and morphisms G . A more flexible notion (though equivalent for many purposes, see [DKS89],[Coo78]) is a homotopy coherent diagram $\underline{X}: BG \rightarrow \text{Top}_*$.

Consider the case of a small category \mathcal{C} *without* a G -action. Then one might define a “ G -equivariant” diagram as a homotopy-coherent diagram $BG \times \mathcal{C} \rightarrow \text{Top}_*$.

For the case of present interest, that is, for a small category \mathcal{C} with G -action, we need a ‘twisted’ version of the above construction. One would then expect a homotopy-coherent diagram $(EG \times \mathcal{C})/G \rightarrow \text{Top}_*$. However, our definition of an external action is not the most general possible; roughly, it is somewhere between a BG commutative (rather than homotopy coherent) diagram and a homotopy coherent diagram $(EG \times \mathcal{C})/G \rightarrow \text{Top}_*$. That is, Definition 3.47 turns out to be equivalent to a functor $\tilde{\mathcal{C}} \rightarrow \text{Top}_*$, using the notation of Remark 3.17, which is homotopy coherent along some faces, and required to be strictly commutative along others (suitably interpreted), as the reader may verify (cf. Definition 3.41).

Proposition 3.50. *Let $F: \mathcal{C} \rightarrow \text{Top}_*$, where \mathcal{C} has an action ψ and F admits an external action, all as in Definition 3.47. Then the homotopy colimit $\text{hocolim} F$ admits a G -action by*

$$g(f_m, \dots, f_1; t_1, \dots, t_m; y) = (\psi_g f_m, \dots, \psi_g f_1; t_1, \dots, t_m; \bar{\psi}_g y).$$

Similarly, if $F: \mathcal{C} \rightarrow K\text{-Top}_$ admits an external action by G , the homotopy colimit in $K\text{-Top}_*$ inherits a $K \times G$ -action by the same formula.*

Proof. This consists of unraveling the Definition 3.39 of homotopy colimits and applying the condition (3.48). We work with the version of the homotopy colimit in which no non-identity isomorphisms appear in the index category (as is possible from our hypotheses

on \mathcal{C}). The K -equivariant version is analogous. One first sees by directly considering Definition 3.39 that G acts on the homotopy colimit (as a set), and the continuity of the G -action in Definition 3.47 implies that the G -action on the homotopy colimit is continuous.

□

Definition 3.51. Let $F_1, F_2: \mathcal{C} \rightarrow K\text{-Top}_*$ be homotopy coherent diagrams, where \mathcal{C} has an action ψ and F_1 and F_2 admit external actions, all as in Definition 3.47. We say that F_1 and F_2 are *externally weakly equivalent* (usually shortened to *weakly equivalent* if the context is clear) if there is a diagram $F_3: \underline{2} \times \mathcal{C} \rightarrow K\text{-Top}_*$, where $\underline{2} \times \mathcal{C}$ is given the product G -action, so that $F_3|_{i \times \mathcal{C}} = F_{i-1}$ and so that F_3 itself has an external action. Furthermore, we require that the maps $F_1(x) \rightarrow F_2(x)$ are weak equivalences of K -spaces for each $x \in \mathcal{C}$.

Lemma 3.52. Let H a subgroup of G , and $F: \mathcal{C} \rightarrow \mathcal{B}_K$ as in Definition 3.47. The H -fixed-point set $(\text{hocolim} F)^H$ of $\text{hocolim} F$ is the homotopy colimit of the homotopy coherent diagram $F^H: \mathcal{C}^H \rightarrow K\text{-Top}_*$ with entries $F^H(u) = F(u)^H$, whose homotopies $F^H(f_i, \dots, f_1)(t_1, \dots, t_{i-1})(y)$ are given by the restriction of the homotopies of F to $F(u)^H$.

Proof. We describe the fixed-point set explicitly. First, by the construction of homotopy colimits, by applying the relations iteratively, each point in $\text{hocolim} F$ may be represented (uniquely) by a tuple $(f_m, \dots, f_1; t_1, \dots, t_m; y)$ for $m \geq 0$, with none of $t_i = 0, 1$. Such a point is in the fixed-point set if and only if

$$(f_m, \dots, f_1) = (hf_m, \dots, hf_1)$$

as tuples in $\text{Hom}(\mathcal{C})$, and $y = hy$.

That is, the f_i must come from the H -fixed arrows, i.e. elements of $\text{Hom}(\mathcal{C}^H)$. Moreover, it is clear that the homotopies are as in the statement of the lemma. \square

3.5.2 Realizations

Lemma 3.53. *Fix a Burnside functor $F: \mathcal{C} \rightarrow \mathcal{B}_K$ where F admits an external action $\bar{\psi}$ by G , for \mathcal{C} a small category so that there is some n for which each sequence of composable non-identity morphisms has length at most n .*

Let \tilde{F}^+ be a \mathfrak{r} -homotopy coherent diagram for F , modeled on a $K \times G$ -representation V , where \mathfrak{r} is V viewed as an orthogonal representation of K . Suppose that

$$g(\tilde{F}_k^+(f_i, \dots, f_1)(t_1, \dots, t_{i-1})(p)) = \tilde{F}_k^+(g(f_i), \dots, g(f_1))(t_1, \dots, t_{i-1})(gp). \quad (3.54)$$

for $p \in B_x / \partial B_x$, and $x \in F(u)$, and finally $g \in G$. Here, g acts on each copy B_x , for $x \in \coprod_{u \in \text{Ob}(\mathcal{C})} F(u)$, of $B(V)$, by using that each B_x is canonically identified with $B(V)$. That is, $g \in G$ acts by $B_x = B(V) \xrightarrow{g} B(V) \xrightarrow{k} B_{gx}$, where $k \in K$ is the label $\sigma(s^{-1}(x))$ for $s^{-1}(x) \in \psi_{g,u}$ where $x \in F(u)$, as in the notation of Definition 3.16.

Then $\text{hocolim} \tilde{F}$ admits a G -action, commuting with its natural K -action, given by

$$(f_m, \dots, f_1; t_1, \dots, t_{m-1}; y) \rightarrow (gf_m, \dots, gf_1; t_1, \dots, t_{m-1}; g(y)).$$

If $F: \underline{2}^{np} \rightarrow \mathcal{B}_K$ is a Burnside functor admitting an external action by \mathbb{Z}_p , with \tilde{F} satisfying the conditions of the Lemma for $\mathcal{C} = \underline{2}^n$, we have that $\|\tilde{F}^+\|$ admits a \mathbb{Z}_p -action, commuting with its natural K -action, as above.

Proof. This follows directly from Proposition 3.50; the last statement is a special case. \square

Note that the condition (3.54) divides up into a family of conditions for each

$1 \leq i \leq n$, which may be checked separately.

Definition 3.55. We call an equivariant refinement \tilde{F}^+ of a Burnside functor $F: \mathcal{C} \rightarrow \mathcal{B}_K$ with a G -external action satisfying (3.54) a *G -coherent refinement* of F .

We next try to build a homotopy coherent diagram satisfying the conditions of Lemma 3.53. The key is to provide a suitable generalization of Proposition 5.2 of [LLS].

Proposition 3.56. [cf. Proposition 5.2 [LLS], Proposition 4.12 [SSS18]] *Let \mathcal{C} be a small category admitting a G -action ψ , in which every sequence of composable non-identity morphisms has length at most n , and let $F: \mathcal{C} \rightarrow \mathcal{B}_K$ be a strictly unitary 2-functor admitting an external G -action.*

1. *For all sufficiently large finite-dimensional representations V of $K \times G$ there exists a G -coherent refinement of F modeled on V .*
2. *There exists some finite-dimensional G -representation W so that for all finite-dimensional representations V of $K \times G$ containing W , any two G -coherent refinements of F modeled on V are weakly equivalent.*
3. *If \tilde{F}_V is a G -coherent spatial refinement of F modeled on V , then the result of suspending each $\tilde{F}_V(u)$ and $\tilde{F}_V(f_n, \dots, f_1)$ by a $K \times G$ representation V' gives a G -coherent spatial refinement of F modeled on $V \oplus V'$.*

Proof. For Item 1, we inductively construct a spatial refinement \tilde{F} .

First, choose representatives a_ω of the orbits of $\text{Hom}(\mathcal{C})$ under the action of G . For each representative a_ω , let $S_\omega \subset G$ be its stabilizer subgroup. For each a_ω , choose a $K \times S_\omega$ -equivariant disk map refining $F(a_\omega)$; such exist by Lemmas 3.35 and 3.36. Then, define the maps associated to each $a \in \text{Hom}(\mathcal{C})$ by, if $ga_\omega = a$, setting $g\tilde{F}(a_\omega)g^{-1} =: \tilde{F}(ga_\omega)$. Here, recall that $g \in G$ acts on $\bigvee_{u \in \mathcal{C}} \bigvee_{x \in F(u)} B(V)_x / \partial$ by $B(V)_x = B(V) \rightarrow^g$

$B(V) \rightarrow^{k(g)} B(V) = B(V)_{gx}$, where $k(g)$ is the label of the arrow $x \rightarrow gx$ (an element of K), so that $\tilde{F}(ga_\omega)$ is a composite (say that a_ω is a correspondence from $F(u)$ to $F(v)$):

$$\begin{aligned} \tilde{F}(gu) &= \bigvee_{gx \in F(gu)} B(V)_{gx} / \partial \rightarrow^{k(g^{-1})g^{-1}} \bigvee_{x \in F(u)} B(V)_x / \partial \xrightarrow{\tilde{F}(a_\omega)} \tilde{F}(v) = \bigvee_{y \in F(v)} B(V)_y / \partial \\ &\quad \rightarrow^{k(g)g} \bigvee_{gy \in F(gv)} B(V)_{gy} = \tilde{F}(gv). \end{aligned}$$

It follows from the construction of $\tilde{F}(a_\omega)$ that $\tilde{F}(a)$ is independent of the choice of g so that $ga_\omega = a$ holds. Let us see that the maps constructed thus satisfy Lemma 3.53 for $i = 1$. Indeed, we need to check

$$g\tilde{F}(f)(p) = \tilde{F}(gf)(gp)$$

for all $g \in G$. By hypothesis, $f = ha_\omega$ for some a_ω . Then $\tilde{F}(ha_\omega)$ is defined by $h\tilde{F}(a_\omega)h^{-1}$, and the $i = 1$ case of (3.54) follows readily, using that $s\tilde{F}(a_\omega)s^{-1} = \tilde{F}(a_\omega)$ for $s \in S_\omega$.

Fix $\ell \geq 1$ and suppose that for any sequence $v_0 \rightarrow^{f_1} \dots \rightarrow^{f_\ell} v_\ell$ of non-identity morphisms we have chosen a map $e_{f_1, \dots, f_\ell} : [0, 1]^{\ell-1} \rightarrow E_{K,V}(\{B_x \mid x \in F(v_0)\}, s_{F(f_\ell \circ \dots \circ f_1)})$, compatible in that:

$$\begin{aligned} e_{f_\ell, \dots, f_1}(t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_{\ell-1}) &= e_{f_\ell, \dots, f_i}(t_{i+1}, \dots, t_{\ell-1}) \circ e_{f_{i-1}, \dots, f_1}(t_1, \dots, t_{i-1}) \\ e_{f_\ell, \dots, f_1}(t_1, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_{\ell-1}) &= e_{f_\ell, \dots, f_i \circ f_{i-1}, \dots, f_1}(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{\ell-1}) \end{aligned}$$

and satisfying the $i = \ell$ condition of Lemma 3.53.

Then, choose representatives a_ω for the orbits of the G -action on the set of all composable tuples $v_0 \rightarrow^{f_1} \dots \rightarrow^{f_{\ell+1}} v_{\ell+1}$ for v_i objects of \mathcal{C} , with stabilizers S_ω as before. Here, G acts on the set of composable tuples by acting diagonally on each of the mor-

phisms in a composable tuple. Then for the induction step, given $a_\omega = (f_1, \dots, f_{\ell+1})$ where $v_0 \xrightarrow{f_1} \dots \xrightarrow{f_{\ell+1}} v_{\ell+1}$ is a composable sequence of arrows, we have a continuous map

$$S^{\ell-1} = \partial([0, 1]^\ell) \rightarrow E_{K,V}(\{B_x \mid x \in F(v_0)\}, s_{F(f_{\ell+1} \circ \dots \circ f_1)})$$

defined by

$$\begin{aligned} (t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_\ell) &\mapsto e_{f_{\ell+1}, \dots, f_{i+1}}(t_{i+1}, \dots, t_\ell) \circ e_{f_i, \dots, f_1}(t_1, \dots, t_{i-1}) \\ (t_1, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_\ell) &\mapsto e_{f_{\ell+1}, \dots, f_{i+1} \circ f_i, \dots, f_1}(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_\ell). \end{aligned} \quad (3.57)$$

By Lemma 3.35, this map extends to a map, call it $e_{f_{\ell+1}, \dots, f_1}$, from $[0, 1]^\ell$, which is $K \times S_\omega$ -equivariant. Define $e_{f'_{\ell+1}, \dots, f'_1}$, for $(f'_{\ell+1}, \dots, f'_1) = ga_\omega$ for some $g \in G$, by $ge_{a_\omega}g^{-1}$. This is well-defined as in the $i = 1$ case (independent of the choice of g for which $(f'_{\ell+1}, \dots, f'_1) = ga_\omega$) and gives that the collection of $e_{(f'_{\ell+1}, \dots, f'_1)}$ thus defined satisfy the $i = \ell + 1$ case of Lemma 3.53.

We have used that external actions respect composition, as in Definition 3.16, in order to see that each $ge_{a_\omega}g^{-1}$ is a family of disk maps refining the composite correspondence $g(f'_{\ell+1} \circ \dots \circ f'_1)$.

By definition, the maps

$$\Phi(e_{f_m, \dots, f_1}): [0, 1]^{m-1} \times \bigvee_{x \in F(v_0)} B(V)_x / \partial B(V)_x \rightarrow \bigvee_{x \in F(v_m)} B(V)_x / \partial B(V)_x$$

assemble to form a homotopy coherent diagram.

Next we address point 2. Fix G -coherent refinements \tilde{F}_i of F , for $i = 0, 1$. It suffices to construct a G -coherent refinement $\tilde{F}_2: \underline{2} \times \mathcal{C} \rightarrow K\text{-Top}_*$ with $\tilde{F}_2|_{\{i\} \times \mathcal{D}} = \tilde{F}_i$, where $F_2: \underline{2} \times \mathcal{C} \rightarrow \mathcal{B}_K$ is two copies of F , along with identity arrows along the $\underline{2}$ -factor. By Item 1 we can construct such \tilde{F}_2 . By construction, each $\tilde{F}_2(\phi_{1,0} \times \text{Id}_u)$ for $u \in \text{Ob}(\mathcal{D})$

will be a homotopy equivalence (where $\phi_{1,0}$ is the unique nonidentity morphism in $\underline{2}$), and by definition we obtain that F_0, F_1 are weakly equivalent.

Item 3 is clear. □

Let us consider the fixed-point set of the homotopy colimit constructed in Lemma 3.53. We state the result only for Burnside functors from the cube category; the result for general \mathcal{C} differs only notationally. Henceforth, we will always view $\underline{2}^{np}$ as a category with \mathbb{Z}_p -action by permuting the coordinates. The fixed-point set is readily identified with a copy of $\iota: \underline{2}^n \rightarrow (\underline{2}^n)^p$, which we call the *canonical embedding* of cube categories.

Call a Burnside functor F with external action *singular* if there exists $u \in \mathcal{C}^H$ and $x \in F(u)^H$ so that ψ_g (the decorated bijection $F(u)^H \rightarrow F(u)^H$) has nontrivial decoration on x . Otherwise, call F *nonsingular*.

Lemma 3.58. *Let \tilde{F} be a \mathbb{Z}_p -coherent refinement of $F: \underline{2}^{np} \rightarrow \mathcal{B}_K$, a nonsingular Burnside functor with external action. For H a subgroup of \mathbb{Z}_p , the H -fixed-point set, $\|F\|^H$, is a $K \times \mathbb{Z}_p/H$ -equivariant realization of the fixed-point Burnside functor F^H . That is, $\|F\|_V^H = \|F^H\|_{V^H}$.*

Proof. By Lemma 3.52, $(\text{hocolim } \tilde{F}^+)^H$ is described explicitly, by restricting to the sub-homotopy-coherent diagram $(\tilde{F}^H)^+: (\underline{2}^{np})_+^H \rightarrow K\text{-Top}_*$ by assigning $\tilde{F}^H(u) = F(u)^H$, with homotopies as in Lemma 3.52. That is,

$$(\text{hocolim } \tilde{F}^+)^H = \text{hocolim}_{(\underline{2}^{np})_+^H} (\tilde{F}^H)^+$$

This sub-homotopy coherent diagram \tilde{F}^H is a K -equivariant refinement of F^H , by unwrapping the definitions. The previous equation then shows that $\|F\|^H = \text{hocolim}(\widetilde{F^H})^+$ for the K -equivariant refinement $\widetilde{F^H} := \tilde{F}^H$. □

Lemma 3.59. *Let $\underline{2}^n \subset (2^n)^p$ be the canonical embedding. Fix a nonsingular Burnside functor $F: (2^n)^p \rightarrow \mathcal{B}_K$ with external action, where F admits an external action lifting the \mathbb{Z}_p action on $(2^n)^p$. We will denote both actions by ψ .*

Let F be a \mathbb{Z}_p -coherent refinement. Then the fixed-point set $(\mathrm{hocolim} F^+)^{\mathbb{Z}_p}$ is $\mathrm{hocolim} \tilde{J}^+$, for \tilde{J} some K -equivariant refinement of $F|_{\underline{2}^n}$.

Proof. This follows immediately from Lemma 3.58. □

We also briefly discuss an equivariant cell decomposition for $\mathrm{hocolim} \tilde{F}^+$ (which will not be needed in the sequel); this depends on some choices. First, choose a fixed $K \times G$ -CW decomposition of $S(V) = B(V)/\partial B(V)$, coming from a $K \times G$ -representation V , as well as $K \times S$ -CW decompositions of each G/H for $H \subset G$ (see [Ada84][§2]).

The most immediate construction is by taking the fine CW structure described above, and dividing it into pieces along the orbits of composable tuples $v_0 \xrightarrow{f_1} \cdots \xrightarrow{f_\ell} v_\ell$ under the action of G . To be more precise, choose representatives $\{a_i\}_{i=1,\dots,N}$ for each orbit (f_1, \dots, f_ℓ) among composable tuples, under the action of G . Let S_i be the stabilizer of a_i . Let v_0 be the starting object of a_i , so that $\tilde{F}(v_0) = \bigvee_{j \in J} S(V_j)$, the sphere associated to V_j , a copy of the $K \times G$ -representation V . Note that S_i acts on the index set J . Choose orbit representatives b_k with stabilizer $S_{i,j}$. Now, $S(V_j)$ becomes a $K \times S_{i,j}$ -representation sphere, and inherits a $K \times S_{i,j}$ -CW decomposition from our above choices. That is, $S(V_j)$ admits a decomposition as a union of cells, running over k , $C_j^k \times (K/K' \times S_{i,j}/S'_{i,j})$ where $K' \subset K$ and $S'_{i,j} \subset S_{i,j}$ are stabilizer subgroups, and C_j^k are ordinary CW cells. Then the corresponding G -CW cells of $(\mathrm{hocolim} \tilde{F}^+)$ are obtained by the G -orbit of the preceding cells. That is, to the pair of orbit representatives (a_i, b_j) , along with a $K \times S_{i,j}$ -cell of $S(V_j)$, say $C_j^{k_0} \times (K/K' \times S_{i,j}/S'_{i,j})$, we obtain an equivariant cell

$$C_j^{k_0} \times (K/K' \times G/S'_{i,j})$$

of $\text{hocolim} \tilde{F}^+$. This gives a reasonably explicit CW decomposition of $\text{hocolim} \tilde{F}^+$, but it is somewhat unwieldy for calculation.

We can now discuss how the realizations of different Burnside functors are related.

Lemma 3.60. *[cf. Lemma 4.15 [SSS18]] A cofibration sequence $J \rightarrow F \rightarrow H$ of functors with external action $\underline{2}^{np} \rightarrow \mathcal{B}_K$, upon realization, induces a cofibration sequence in $(K \times \mathbb{Z}_p)\text{-Top}_*$. In general, any external natural transformation $\eta: F_1 \rightarrow F_0$ of Burnside functors $\underline{2}^{np} \rightarrow \mathcal{B}_K$ induces a $(K \times \mathbb{Z}_p)$ -equivariant map on the realizations, well-defined up to K -equivariant homotopy.*

Proof. The proof is parallel to that of Lemma 4.15 of [SSS18], which produces a K -equivariant map of realizations as a Puppe map. We will need some of the details in the proof of Lemma 3.61, so we go over the argument.

If $\eta: \underline{2}^{np+1} \rightarrow \mathcal{B}_K$ is the natural transformation, then $(F_0)_{\iota_0}$ is a subfunctor and $(F_1)_{\iota_1}$ is the corresponding quotient functor, where $\iota_i: \underline{2}^{np} \rightarrow \underline{2}^{np+1}$ is the face inclusion to $\{i\} \times \underline{2}^{np}$. We obtain a cofibration sequence

$$\|(F_0)_{\iota_0}\| \rightarrow \|\eta\| \rightarrow \|(F_1)_{\iota_1}\|.$$

However, $\|(F_0)_{\iota_0}\| = \|F_0\|$, while $\|(F_1)_{\iota_1}\| = \Sigma\|F_1\|$ since $\|F_1\|$ is constructed as a homotopy colimit over $\underline{2}_+^n$, while $\|(F_1)_{\iota_1}\|$ is constructed as a homotopy colimit over $\underline{2}_+^{n+1}$.

Therefore, the Puppe map

$$\|(F_1)_{\iota_1}\|_V = \Sigma\|F_1\|_V = \|F_1\|_{V \oplus \mathbb{R}} \rightarrow \Sigma\|(F_0)_{\iota_0}\|_V = \Sigma\|F_0\|_V = \|F_0\|_{V \oplus \mathbb{R}}$$

is the required map. To see that the map is also \mathbb{Z}_p -equivariant, we use that, under the hypothesis of Lemma 3.60, the cofibration sequence itself is \mathbb{Z}_p -equivariant, from which the Puppe map can be chosen to be \mathbb{Z}_p -equivariant.

□

Write η_* for the map $||F_1||_V \rightarrow ||F_0||_V$ as in Lemma 3.60.

Lemma 3.61. *[cf. Proposition 4.16 [SSS18]] If $F: \underline{2}^{np} \rightarrow \mathcal{B}_K$ is a Burnside functor with \mathbb{Z}_p -external action, then its shifted reduced coarse cellular complex $\tilde{C}_{\text{cell}}(||F||_V)[- \dim V]$ is isomorphic to the totalization $\text{Tot}(F)$, with the cells mapping to the corresponding generators. If $\eta: F_1 \rightarrow F_0$ is an external natural transformation, then the map $\eta_*: ||F_1||_V \rightarrow ||F_0||_V$ is homotopic to a map which is cellular with respect to the coarse CW structure, and such that the induced cellular map on the coarse structure agrees with $\text{Tot}(\eta)$. Moreover, if F is nonsingular, the restriction to fixed points $\eta_*|_H: ||F_1||_V^H \rightarrow ||F_0||_V^H$, for H a subgroup of \mathbb{Z}_p , is K -equivariantly homotopic to $\eta_*^H: ||F_1^H||_{V^H} \rightarrow ||F_0^H||_{V^H}$, the map of realizations induced by the H -fixed-point functor $\eta^H: F_1^H \rightarrow F_0^H$. Here we have used $||F_i||_V^H = ||F_i^H||_{V^H}$, for suitable realizations, by Lemma 3.58. Also, η_*^H is K -equivariantly homotopic to a cellular map on the coarse CW structures on $||F_i||_V^H$. Finally, the induced cellular chain map on the H -fixed points, in the coarse CW structure of the fixed points, is $\text{Tot}(\eta^H)$.*

Recall that neither the coarse nor fine CW structures need be equivariant CW structures.

Proof. The first claim is just Proposition 3.45, and does not involve the external action. The map η_* constructed in Lemma 3.60 is not necessarily cellular, but by (ho-1), the homotopy-type of η_* is (not necessarily \mathbb{Z}_p -equivariantly) K -equivariantly well-defined. However, the map constructed in Proposition 3.45 is cellular, with induced cellular chain map $\text{Tot}(\eta)$. This establishes the claim that η_* is K -equivariantly homotopic to a cellular map with the appropriate induced map on cellular chain complexes.

Finally, we address the induced map on fixed-point sets. The natural transformation η restricts to a natural transformation of Burnside functors $\eta^H: F_1^H \rightarrow F_0^H$. The realization of η^H , along with some additional choices, defines a map $\eta_*^H: ||F_1^H|| \rightarrow ||F_0^H||$. On the

other hand, we have that $\eta_*|_H$ is homotopic to η_*^H , by construction. Now, applying Proposition 3.45 exactly as in the non-fixed-point case, we obtain that η_*^H is homotopic to a cellular map, and that cellular map has induced map on chain complexes given by $\text{Tot}(\eta^H)$. This completes the proof. \square

Remark 3.62. The Lemmas 3.60 and 3.61 are unsatisfactory in several regards. First, we expect that it should be possible to choose the map in Lemma 3.60 to be both cellular for the coarse CW structure, and \mathbb{Z}_p -equivariant. Second, we expect for the resulting map η_* that the restricted maps $\eta_*|_H$, as in Lemma 3.61, are cellular with respect to the coarse CW structure on the H -fixed-point sets. These issues are not especially disturbing, as our object is to understand the homotopy colimit, not necessarily the coarse CW structures (which, indeed, are not equivariant CW structures to begin with), although use of the coarse CW structure makes it easier to translate statements about Khovanov homology to Khovanov spectra.

Another problem is that the map constructed in Lemma 3.60 is not guaranteed to be well-defined up to \mathbb{Z}_p -equivariant homotopy. To establish any form of ‘naturality’ for equivariant Khovanov spectra, this well-definedness would be necessary.

Furthermore, in Lemma 3.61, it should be possible to identify $\widetilde{C}_{\text{cell}}(\|F\|_V)[- \dim V]$ with the totalization $\text{Tot}(F)$ as \mathbb{Z}_p -chain complexes.

Note that Lemma 3.61 applies for any spatial realization of a Burnside functor with external action; in turn we use it to prove:

Lemma 3.63. *Let $F: \underline{2}^{np} \rightarrow \mathcal{B}_K$ be a nonsingular Burnside functor with external action by \mathbb{Z}_p . For V sufficiently large, $\|F\|_V$ is well-defined up to homotopy-equivalence in $(K \times \mathbb{Z}_p)\text{-Top}_*$.*

Proof. Say that there exist refinements \widetilde{F}_0 and \widetilde{F}_1 . By Proposition 3.56(2), there is a homotopy coherent diagram $\widetilde{\eta}: \underline{2} \times \underline{2}^{np} \rightarrow K\text{-Top}_*$ so that $\widetilde{\eta}|_{i \times \underline{2}^{np}} = \widetilde{F}_i$. By construction,

$\text{Tot}(\eta^H)$, for any subgroup $H \subset G$, is the identity. By Lemma 3.61, the connecting map $||\tilde{F}_1|| \rightarrow ||\tilde{F}_0||$ is a $K \times \mathbb{Z}_p$ -equivariant homotopy equivalence (since it induces chain homotopy equivalences on all fixed-point sets), from which the Lemma follows. \square

In order to describe the relationship between realizations of externally stably equivalent Burnside functors, we need a further object. Let $J_p: \underline{2}^p \rightarrow \mathcal{B}_K$ be the Burnside functor (with external action by \mathbb{Z}_p) with $J_p(1^p)$ a 1-element set, and $J_p(v) = \emptyset$ for $v \neq 1^p$.

Lemma 3.64. *The realization of J_p satisfies $||J_p||_V = \Sigma^V(\mathbb{R}(\mathbb{Z}_p))^+$.*

Proof. This is an exercise in the definitions. \square

We also need a simple fact about indexing categories:

Lemma 3.65. *The natural \mathbb{Z}_p -equivariant map $\underline{2}_+^p \times \underline{2}_+^{np} \rightarrow \underline{2}_+^{(n+1)p}$ is homotopy cofinal.*

Proof. This is similar to the proof of [LLS][Lemma 4.18]. \square

Proposition 3.66. *An external K -equivariant stable equivalence $(E_1, W_1) \rightarrow (E_2, W_2)$ of stable nonsingular functors $(E_1: \underline{2}^{n_1 p} \rightarrow \mathcal{B}_K, W_1)$ and $(E_2: \underline{2}^{n_2 p} \rightarrow \mathcal{B}_K, W_2)$ induces a $K \times \mathbb{Z}_p$ -equivariant homotopy equivalence $|\Sigma^{W_1} E_1| \rightarrow |\Sigma^{W_2} E_2|$.*

Proof. We need only check that the operations (1) and (2) of Definition 3.26 induce equivariant homotopy equivalences.

For operation (1), say we have a natural transformation $F_1 \rightarrow F_2$ of Burnside functors with external action. Associated to a natural transformation with external action, there is a map $||F_1||_V \rightarrow ||F_2||_V$ for any realizations, for V sufficiently large, by Lemma 3.60. By Lemma 3.61, the resulting (equivariant) map is a homotopy equivalence $||F_1^H|| \rightarrow ||F_2^H||$ for all H (having applied the Whitehead theorem on each fixed-point set). A similar argument handles subgroups $S \subset K \times \mathbb{Z}_p$ with nontrivial projection to K . By the

G -Whitehead theorem, we have that F_1 and F_2 are $(K \times \mathbb{Z}_p)$ -equivariantly homotopy equivalent.

We next deal with operation (2). It will suffice to show that for the face inclusion $\iota: \underline{2}^{np} \rightarrow \underline{2}^{(n+1)p}$ and a Burnside functor $F: \underline{2}^{np} \rightarrow \mathcal{B}_K$, that $\Sigma^{\mathbb{R}(\mathbb{Z}_p)}||F||$ is (equivariantly) homotopy equivalent to $||F_\iota||$. We will check this using the relationship of homotopy colimits to smash products.

First, observe by (ho-4) that we have a natural weak equivalence:

$$(\mathrm{hocolim}_{\underline{2}_+^p} \tilde{J}^+) \wedge (\mathrm{hocolim}_{\underline{2}_+^{np}} \tilde{F}^+) \rightarrow \mathrm{hocolim}_{\underline{2}_+^p \times \underline{2}_+^{np}} (\tilde{J}^+ \wedge \tilde{F}^+). \quad (3.67)$$

We must check first that this map is \mathbb{Z}_p -equivariant. To do so, we would like to use naturality of the map in (ho-4). In order to use that naturality, we need to use the external action to generate homomorphisms of homotopy coherent diagrams.

Choose a generator $g \in \mathbb{Z}_p$. Let $\mathbf{F}_{g^{-1}}: \underline{2}_+^{np} \rightarrow \underline{2}_+^{np}$ be the action of g^{-1} on $\underline{2}_+^{np}$. We consider the pullback homotopy coherent diagram $\mathbf{F}_{g^{-1}}(\tilde{F}^+)$. The external action of \mathbb{Z}_p on \tilde{F}^+ defines a homomorphism of homotopy coherent diagrams $\mathbf{F}_{g^{-1}}(\tilde{F}^+) \rightarrow \tilde{F}^+$. We then obtain a well-defined K -equivariant map, by (ho-2),

$$\mathrm{hocolim} \mathbf{F}_{g^{-1}}(\tilde{F}^+) \rightarrow \mathrm{hocolim} \tilde{F}^+.$$

Note that $\mathrm{hocolim} \mathbf{F}_{g^{-1}}(\tilde{F}^+)$ is not identical to $\mathrm{hocolim} \tilde{F}^+$, however, there is a natural homeomorphism $\mathrm{hocolim} \mathbf{F}_{g^{-1}}(\tilde{F}^+) \rightarrow \mathrm{hocolim} \tilde{F}^+$, essentially by relabeling. All of this discussion applies equally well, replacing \tilde{F}^+ with \tilde{J}^+ or $\tilde{F}^+ \wedge \tilde{J}^+$. In fact, we have a commutative diagram:

$$\begin{array}{ccc}
(\mathrm{hocolim}_{\underline{2}_+^p}(\mathbf{F}_{g^{-1}}\tilde{J}^+)) \wedge (\mathrm{hocolim}_{\underline{2}_+^{np}}(\mathbf{F}_{g^{-1}}\tilde{F}^+)) & \rightarrow & \mathrm{hocolim}_{\underline{2}_+^p \times \underline{2}_+^{np}}(\mathbf{F}_{g^{-1}}(\tilde{J}^+ \wedge \tilde{F}^+)) \\
\downarrow & & \downarrow \\
(\mathrm{hocolim}_{\underline{2}_+^p}\tilde{J}^+) \wedge (\mathrm{hocolim}_{\underline{2}_+^{np}}\tilde{F}^+) & \longrightarrow & \mathrm{hocolim}_{\underline{2}_+^p \times \underline{2}_+^{np}}(\tilde{J}^+ \wedge \tilde{F}^+)
\end{array}$$

Moreover, the \mathbb{Z}_p -action on $\mathrm{hocolim}\tilde{F}^+$ factors nicely, in that we have a commutative diagram:

$$\begin{array}{ccc}
\mathrm{hocolim}_{\underline{2}_+^{np}}(\mathbf{F}_{g^{-1}}\tilde{F}^+) & & \\
\downarrow g & \searrow & \mathrm{hocolim}_{\underline{2}_+^{np}}\tilde{F}^+ \\
& \swarrow g & \\
\mathrm{hocolim}_{\underline{2}_+^{np}}\tilde{F}^+ & &
\end{array}$$

where the vertical arrow is the map induced by the homomorphism, and the diagonal arrow labeled by g is as in the definition of the \mathbb{Z}_p action on $\mathrm{hocolim}\tilde{F}^+$. The analogous diagrams for $(\mathrm{hocolim}_{\underline{2}_+^p}\tilde{J}^+) \wedge (\mathrm{hocolim}_{\underline{2}_+^{np}}\tilde{F}^+)$ and $\mathrm{hocolim}_{\underline{2}_+^p \times \underline{2}_+^{np}}(\tilde{J}^+ \wedge \tilde{F}^+)$ also commute. Using the above commutative square, and the naturality of (ho-4) with respect to homomorphisms, we see that (3.67) is \mathbb{Z}_p -equivariant.

We note that $\tilde{J}^+ \wedge \tilde{F}^+$ is the pullback of some G -coherent spatial refinement $\widetilde{J \times F}^+$ under $L: \underline{2}_+^p \times \underline{2}_+^{np} \rightarrow \underline{2}_+^{(n+1)p}$, as follows from the definitions.

Moreover, it is immediate from the definitions that $J \times F = F_L$.

Using Lemma 3.65 and (ho-3), we have a homotopy-equivalence

$$\mathrm{hocolim}_{\underline{2}_+^p \times \underline{2}_+^{np}}(\tilde{J}^+ \wedge \tilde{F}^+) \simeq \mathrm{hocolim}_{\underline{2}_+^{(n+1)p}}(\widetilde{J \times F}^+).$$

Moreover, this homotopy-equivalence is once again equivariant with respect to the \mathbb{Z}_p -action, because it is natural in the involved diagrams. For each subgroup $H \subset \mathbb{Z}_p$, we obtain a similar map on diagrams of H -fixed-point sets. However, the same hypotheses

we have used to this point apply to the H -fixed-point sets, since they come from refinements of the Burnside functor F^H , according to Lemma 3.58. That is, the map on H -fixed-point sets is also a homotopy equivalence, and by the G -Whitehead Theorem, we have obtained:

$$(\mathrm{hocolim}_{\underline{2}_+^p} \tilde{J}^+) \wedge (\mathrm{hocolim}_{\underline{2}_+^{np}} \tilde{F}^+) \simeq \mathrm{hocolim}_{\underline{2}_+^{(n+1)p}} (\widetilde{J \times F})^+,$$

equivariantly. Applying Lemma 3.64, the result follows. \square

Remark 3.68. Expanding on Remark 3.62, we expect that there is a form of Proposition 3.66 so that the induced map is well-defined up to $K \times \mathbb{Z}_p$ -equivariant stable homotopy equivalence.

3.6 Applications to Khovanov spectra and homology

In this section, we recall the definition and main properties of Khovanov spectra from [LLS], as well as the generalization of the Lawson-Lipshitz-Sarkar construction to the odd Khovanov case [SSS18].

Fix a link L with diagram D , from which we obtain the Khovanov functor $\mathfrak{F}_e(D): \underline{2}^n \rightarrow \mathbb{Z}\text{-Mod}$ and the odd Khovanov functor $\mathfrak{F}_o(D): \underline{2}^n \rightarrow \mathbb{Z}\text{-Mod}$; we will often omit the diagram D from the notation where it is clear from context. In [LLS], Lawson-Lipshitz-Sarkar extended $\mathfrak{F}_e: \underline{2}^n \rightarrow \mathbb{Z}\text{-Mod}$ to $\mathcal{KH}: \underline{2}^n \rightarrow \mathcal{B}$:

$$\begin{array}{ccc} & & \mathcal{B} \\ & \nearrow \kappa\mathcal{H} & \downarrow \\ \underline{2}^n & \xrightarrow{\mathfrak{F}_e} & \mathbb{Z}\text{-Mod} \end{array}$$

and in [SSS18], $\mathfrak{F}_o(L)$ was extended to $\mathcal{KHO}: \underline{2}^n \rightarrow \mathcal{B}_{\mathbb{Z}_2}$, so that for $\mathfrak{d} = 0: \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$, $\mathrm{Tot}_{\mathfrak{d}}(\mathcal{KHO}) = \mathfrak{F}_e$ and for $\mathfrak{d} = \mathrm{Id}$, $\mathrm{Tot}_{\mathfrak{d}}(\mathcal{KHO}) = \mathfrak{F}_o$. In [SSS18], it was shown that

the equivariant stable-equivalence class of $\mathcal{KHO}(D)$ is an invariant of the link L . From \mathcal{KHO} , one can construct an infinite family $\mathcal{X}_n(L)$ of Khovanov spaces (or spectra), well-defined up to stable homotopy.

Once we have recalled these definitions, we will see in Section 3.6.2 that the machinery of Sections 3.3-3.5 applies to the Khovanov-Burnside functors \mathcal{KHO} and \mathcal{KH} . That is, we will show that \mathcal{KH} and \mathcal{KHO} , as well as their annular analogs, admit external actions in various settings. This is largely, but not entirely, formal. In Section 3.6.3, we will show that the fixed-point functors of these Khovanov-Burnside functors agree with certain *annular Khovanov-Burnside functors*. This section is not formal, and relies on understanding the relationship between resolution configurations in the periodic link and the quotient link; this becomes particularly complicated in the odd case. In Section 3.6.4, we show that the action is well-defined; this section is largely formal once an understanding of the fixed-point functors is dealt with (one can also obtain the results of this section without knowing the fixed-point functor explicitly, but it is somewhat easier with the results of Section 3.6.3 in hand). Here we also wrap up the construction of space-level invariants, proving Theorem 3.3 using the tools from Section 3.5. We end with some spectral sequences in Section 3.6.5, and some questions in Section 3.6.6.

3.6.1 The Khovanov-Burnside functor

The purpose of this section is to explicitly describe the various Burnside functors we will use (cf. [LLS17a] §6).

We start by recalling the construction of the functor $\mathcal{KHO}(D)$, for D a diagram of an oriented link L , with n ordered crossings, and a choice of orientation of crossings, as well as a choice of edge assignment as in Section 3.2.4, and finally an ordering of the circles of each resolution. Following Lemma 3.14, it suffices to define it on objects, edges

$\phi_{u,v}$ with $u \geq_1 v$, and across two-dimensional faces of the cube $\underline{2}^n$. On objects, set

$$\mathcal{KHO}(u) = Kg(u).$$

For each edge $u \geq_1 v$ in $\underline{2}^n$, and each element $y \in \mathcal{KHO}(v)$, write

$$\mathfrak{F}_o(\phi_{v,u}^{\text{op}})(y) = \sum_{x \in \mathcal{KHO}(u)} \epsilon_{x,y} x.$$

Note each $\epsilon_{x,y} \in \{-1, 0, 1\}$. Define

$$\mathcal{KHO}(\phi_{u,v}) = \{(y, x) \in \mathcal{KHO}(v) \times \mathcal{KHO}(u) \mid \epsilon_{x,y} = \pm 1\},$$

where the sign on elements of $\mathcal{KHO}(\phi_{u,v})$ is given by $\epsilon_{x,y}$ of the pair, and the source and target maps are the natural ones.

We need only define the 2-morphisms across 2-dimensional faces. In fact, there is a unique choice of 2-morphisms compatible with the preceding data. To be more specific, for any 2-dimensional face $u \geq_1 v, v' \geq_1 w$, and any pair $(x, y) \in \mathcal{KHO}(u) \times \mathcal{KHO}(w)$, there is a unique bijection between

$$A_{x,y} := s^{-1}(x) \cap t^{-1}(y) \subset \mathcal{KHO}(\phi_{v,w}) \times_{\mathcal{KHO}(v)} \mathcal{KHO}(\phi_{u,v})$$

and

$$A'_{x,y} := s^{-1}(x) \cap t^{-1}(y) \subset \mathcal{KHO}(\phi_{v',w}) \times_{\mathcal{KHO}(v')} \mathcal{KHO}(\phi_{u,v'})$$

that preserves the signs. (That is, the signed sets $A_{x,y}, A'_{x,y}$ both have at most one element of any given sign). Indeed, the only resolution configurations for which $A_{x,y}$ has more than one element are the ladybug configurations. The unique sign-preserving matching turns out to be the right ladybug matching of [LS14] for a type X edge assign-

ment, and is the left ladybug matching for a type Y edge assignment. This completes the description of a strictly unitary lax 2-functor $\mathcal{KHO}_0(D)$ associated to the data as above. We call the identification (for any x, y) of sets $A_{x,y}$ and $A'_{x,y}$ above the *ladybug matching*. Recall also that we work with *stable* Burnside functors; that is, pairs of a Burnside-functor and an integer. We define the (odd) *Khovanov-Burnside functor* by $\mathcal{KHO} = (\mathcal{KHO}_0, -n_-)$. It follows from the construction that we may write \mathcal{KHO} as a sum over quantum gradings: $\mathcal{KHO} = \coprod_j \mathcal{KHO}^j$.

Recall that \mathcal{KHO} is a link invariant:

Theorem 3.69 (Theorem 1.7 [SSS18]). *The equivariant stable equivalence class of the stable functor \mathcal{KHO} is independent of the choices in its construction, and is a link invariant. Let $\widetilde{\mathcal{KHO}}_{j,n}$ be a spatial refinement of \mathcal{KHO}^j in sufficiently high dimension, modeled on $\tilde{\mathbb{R}}^n$. Then the stable homotopy-type of the spatial realization $\mathcal{X}_n^j = \|\widetilde{\mathcal{KHO}}_{j,n}^+\|$ is a link invariant. Moreover, there is a CW structure on \mathcal{X}_n^j for which the reduced cellular chain complex $\tilde{C}_{\text{cell}}^*(\mathcal{X}_n^j) = Kc_o^j(L; \mathbb{Z}) = \text{Tot}_{\text{Id}}(\mathcal{KHO}^j)^*$ if n odd, or $Kc^j(L; \mathbb{Z}) = \text{Tot}_{\partial=0}(\mathcal{KHO}^j)^*$ if n is even.*

Let $\mathcal{KH} = \mathcal{F}(\mathcal{KHO})$ denote the Burnside functor obtained by forgetting signs; call this the *even Khovanov-Burnside functor*; it agrees with the construction of [LLS]. We illustrate an example Khovanov-Burnside functor in Figure 3.3.

Next, we address the construction of the *annular Khovanov-Burnside functor* $\mathcal{AKHO}(D) = \coprod_{j,k} \mathcal{AKHO}^{j,k}(D): \underline{2}^n \rightarrow \mathcal{B}_{\mathbb{Z}_2}$ associated to a diagram D of an annular link L , along with an ordering of the n crossings, an orientation of the crossings, a choice of edge assignment, and an ordering of circles at each resolution. We define, for $u \in \underline{2}^n$,

$$\mathcal{AKHO}^{j,k}(u) = Kg^{j,k}(u).$$

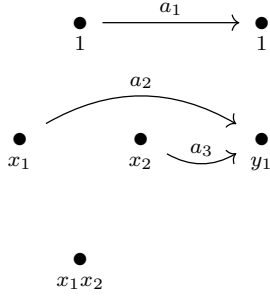


Figure 3.3: An example Burnside functor $F: \underline{2}^1 \rightarrow \mathcal{B}_{\mathbb{Z}_2}$. We visualize elements of $F(1), F(0)$ as dots, and regard the morphism $F(\phi)$ as a collection of arrows. Here, we let $F(1) = \{1, x_1, x_2, x_1x_2\}$, the set of Khovanov generators associated to a resolution configuration of two circles, and $F(0) = \{1, y_1\}$, the set of Khovanov generators associated to a single circle. Set $F(\phi_{1,0}) = \{a_1, a_2, a_3\}$; $s(a_i)$ is given by the tail of the arrow a_i , and $t(a_i)$ is given by the head of the arrow a_i . This is the Khovanov-Burnside functor associated to two circles merging to a single circle.

For each edge $u \geqslant_1 v$ in $\underline{2}^n$, and each element $y \in \mathcal{AKHO}(v)$, write

$$\mathfrak{F}_{\text{Ann}_o}(\phi_{v,u}^{\text{op}})(y) = \sum_{x \in \mathcal{AKHO}(u)} \epsilon_{x,y} x.$$

Define

$$\mathcal{AKHO}(\phi_{u,v}) = \{(y, x) \in \mathcal{AKHO}(v) \times \mathcal{AKHO}(u) \mid \epsilon_{x,y} = \pm 1\},$$

where the sign on elements of $\mathcal{AKHO}(\phi_{u,v})$ is given by $\epsilon_{x,y}$ of the pair, and the source and target maps are the natural ones. The matching along 2-dimensional faces is obtained from that of \mathcal{KHO} , and the formal desuspension of \mathcal{AKHO} is also inherited from \mathcal{KHO} .

We have the following theorem:

Theorem 3.70. *The equivariant stable equivalence class of the functor $\mathcal{AKHO}(D)$ is independent of the choices involved in its construction, and is an invariant of the annular link L . Let $\widetilde{\mathcal{AKHO}}_{j,k,n}$ be a spatial refinement of $\mathcal{AKHO}^{j,k}$ in sufficiently high dimension, modeled on $\tilde{\mathbb{R}}^n$. Then the stable homotopy-type of the spatial realization $\mathcal{AKH}_n^{j,k} = ||\widetilde{\mathcal{AKHO}}_{j,k,n}^+||$ is a link invariant. Moreover, there is a CW structure on $||\widetilde{\mathcal{AKHO}}_{j,k,n}^+||$ for which the reduced cellular chain complex $\tilde{C}_{\text{cell}}^*(\mathcal{AKH}_n^{j,k}) =$*

$AKc_o^j(L; \mathbb{Z}) = \text{Tot}_{\text{Id}}(\mathcal{AKHO}^{j,k})^*$ if n odd, or $AKc^j(L; \mathbb{Z}) = \text{Tot}_{\mathfrak{d}=0}(\mathcal{AKHO}^{j,k})^*$ if n is even.

Proof. This follows from keeping track of the annular gradings in the invariance proof of the equivariant stable equivalence class of \mathcal{KHO} . \square

We write $\mathcal{AKH}: \underline{\mathbb{Z}}^n \rightarrow \mathcal{B}$ for the *even annular Khovanov-Burnside functor*, obtained from \mathcal{AKHO} by forgetting the sign.

3.6.2 Equivariant Khovanov-Burnside functors

In this section, we apply the machinery from Sections 3.3-3.5 to construct Burnside functors with external action. We first outline the notation used in this section. Let $p \geq 1$ be an integer, and consider a p -periodic link \tilde{L} with (annular) periodic diagram \tilde{D} . The action by \mathbb{Z}_p on \tilde{L} and \tilde{D} are both denoted ψ . The quotient link $L = \tilde{L}/\psi$ has (annular) diagram $D = \tilde{D}/\psi$, with, say, n crossings. We refer to information relating to \tilde{L} as ‘upstairs’ and information relating to the quotient L as ‘downstairs.’

Theorem 3.71. *Let \tilde{L} be a p -periodic link. Then there is a natural \mathbb{Z}_p -external action on $\mathcal{AKH}(\tilde{L})$ and $\mathcal{KH}(\tilde{L})$, whose external stable equivalence class is an invariant of the equivariant isotopy type of the link \tilde{L} . If p is odd, then there is a natural \mathbb{Z}_p -external action on $\mathcal{AKHO}(\tilde{L})$ and $\mathcal{KHO}(\tilde{L})$, whose external stable equivalence class is an invariant of the equivariant isotopy type of the link \tilde{L} .*

We will prove this theorem over the course of the next few sections. We start with the construction.

Proposition 3.72. *Let \tilde{D} be a p -periodic link diagram. There is a natural \mathbb{Z}_p -external action on $\mathcal{AKH}(\tilde{D})$ and $\mathcal{KH}(\tilde{D})$.*

Proof. Recall from Section 3.2.6 that \mathbb{Z}_p acts on $\coprod_{u \in \underline{2}^{np}} Kg^{j,k}(u)$ for any $j, k \in \mathbb{Z}$. For $u \geq_1 v$, it is easy to check that there are natural bijections $\psi: \mathcal{AKH}(\phi_{u,v}) \rightarrow \mathcal{AKH}(\phi_{\psi u, \psi v})$ and $\psi: \mathcal{KH}(\phi_{u,v}) \rightarrow \mathcal{KH}(\phi_{\psi u, \psi v})$. We are almost in the situation of Lemma 3.19: in order to apply that lemma, we need only show that the \mathbb{Z}_p -action respects the ladybug matching. However, this is also essentially automatic; let us see how formal properties of the ladybug matching guarantee this. First of all, we need only consider squares $u \geq_1 v, v' \geq_1 w$ in $\underline{2}^{np}$ so that $\mathcal{KH}(\phi_{u,v})$ (or $\mathcal{AKH}(\phi_{u,v})$) has two elements, otherwise the diagram in the hypotheses (E-2 ') of Lemma 3.19 is automatically commutative. That is, we may assume the resolution configuration associated to $u \geq_2 w$ is a ladybug configuration.

Then the arrows from the action in (E-2 ') are obtained from the maps

$$Kg(u) \rightarrow Kg(\psi u),$$

(similarly for v, v', w), obtained by rotating the resolution D_u to $D_{\psi u}$ (using that $\mathcal{KH}(\phi_{u,v})$, etc., is a subset of the product $\mathcal{KH}(u) \times \mathcal{KH}(v)$). Finally, the ladybug matching is an invariant of planar isotopy, as in [LS14, Lemma 5.8], and so the diagram commutes. Lemma 3.19 then implies that there is a natural \mathbb{Z}_p -external action on $\mathcal{AKH}(\tilde{D})$ and $\mathcal{KH}(\tilde{D})$, as needed. \square

We next generalize this to the odd case. We will need an auxiliary lemma.

Lemma 3.73. *Say p is odd. Let $C_{\text{cell}}^{\mathbb{Z}_p, *}([0, 1]^{np}; \mathbb{F}_2)$ be the subcomplex of $C_{\text{cell}}([0, 1]^{np}; \mathbb{F}_2)$ consisting of \mathbb{Z}_p -invariant chains, with respect to the product cell structure on $[0, 1]^{np}$. Then $H^2(C_{\text{cell}}^{\mathbb{Z}_p}([0, 1]^{np}; \mathbb{F}_2)) = H^1(C_{\text{cell}}^{\mathbb{Z}_p}([0, 1]^{np}; \mathbb{F}_2)) = 0$.*

Proof. Say $c \in C_{\text{cell}}^{\mathbb{Z}_p, 1}([0, 1]^{np}; \mathbb{F}_2)$ has $\delta c = 0$. Now, $c = \delta e$ for some $e \in C_{\text{cell}}^0([0, 1]^{np}; \mathbb{F}_2)$. Then $(1 + \psi)\delta e = 0$, from which we see that $(1 + \psi)e$ is a cocycle. However, the only cocycles in $C_{\text{cell}}^0([0, 1]^{np}; \mathbb{F}_2)$ are the constant cochains evaluating to 0 or 1 on all vertices

of $[0, 1]^{np}$. The cochain evaluating to 1 is not in the image of $1 + \psi$, since the image of $1 + \psi$ is characterized as those cochains so that on each \mathbb{Z}_p -orbit, the sum of evaluation over the orbit is 0. Thus $(1 + \psi)e = 0$, and so c is the boundary of an invariant cochain, as needed.

Next, say $c \in C_{\text{cell}}^{\mathbb{Z}_p, 2}([0, 1]^{np}; \mathbb{F}_2)$ with $\delta c = 0$. Say $c = \delta e$ for $e \in C_{\text{cell}}^1([0, 1]^{np}; \mathbb{F}_2)$. As before $(1 + \psi)e$ is a cocycle, and in particular, say $\delta f = (1 + \psi)e$. Then $\delta(1 + \psi + \cdots + \psi^{p-1})f = 0$. That is, $(1 + \psi + \cdots + \psi^{p-1})f$ is either the constant 0-cocycle or the constant 1-cocycle. Since p is odd, we obtain that, in the former case, f must vanish on invariant vertices of $[0, 1]^{np}$, and in the latter case, f evaluates to 1 on the invariant vertices. However, adding the nontrivial cocycle to f still produces a cochain f' so that $\delta f' = (1 + \psi)e$, and so we may assume that f vanishes on all the invariant vertices of $[0, 1]^{np}$, and that $(1 + \psi + \cdots + \psi^{p-1})f = 0$. That is, f lives in a free \mathbb{Z}_p -submodule of $C_{\text{cell}}^0([0, 1]^{np}; \mathbb{F}_2)$, and using $(1 + \psi + \cdots + \psi^{p-1})f = 0$, it follows that $f = (1 + \psi)g$ for some $g \in C_{\text{cell}}^0([0, 1]^{np}; \mathbb{F}_2)$. Then $\delta g = e + e'$ for some e' in the image of multiplication by $(1 + \psi + \cdots + \psi^{p-1})$, since \mathbb{Z}_p acts freely on $C_{\text{cell}}^1([0, 1]^{np}; \mathbb{F}_2)$. Then, since $\delta^2 = 0$, we have $\delta e' = c$. Finally, the image of $(1 + \psi + \cdots + \psi^{p-1})$ on $C_{\text{cell}}^1([0, 1]^{np}; \mathbb{F}_2)$ is equal to the set of invariant cochains in degree 1, so c is the boundary of an invariant cochain, as needed. \square

Proposition 3.74. *Say p is odd. Let \tilde{D} be a p -periodic link diagram. Then there is a natural \mathbb{Z}_p -external action on $\mathcal{AKHO}(\tilde{D})$ and $\mathcal{KHO}(\tilde{D})$. Moreover, this \mathbb{Z}_p external action is nonsingular.*

Proof. We begin by choosing an equivariant orientation of crossings for \tilde{D} , by which we mean that for each orbit of the np crossings of \tilde{D} under the action of \mathbb{Z}_p , we choose a representative crossing, orient it, and then use the \mathbb{Z}_p -action to define an orientation of crossings for all crossings in the same orbit.

Next, we need to show that there exists an equivariant edge assignment. By this,

we mean that the function ϵ as in Section 3.2.4 can be chosen so that $\epsilon_{v,u} = \epsilon_{\psi v, \psi u}$. For $p = 2$, this is not generally possible, as the reader may confirm by drawing the usual picture of the Hopf link. However, recall that an edge assignment amounts to the choice of an element $\epsilon \in C_{\text{cell}}^1([0, 1]^{np}; \mathbb{F}_2)$ with coboundary $\delta\epsilon = \Omega(\tilde{D})$ (tacitly identifying $\mathbb{Z}_2 = \{\pm 1\}$ with \mathbb{F}_2). We first observe that $\Omega(\tilde{D})$ is \mathbb{Z}_p -equivariant, since the odd resolution configuration $C_{u,w}$ for $u \geq_2 w$ is planar isotopic to the odd resolution configuration $C_{\psi u, \psi w}$, and since $\Omega(\tilde{D})_{u,w}$ is determined by the isotopy type of $C_{u,w}$ for each $u \geq_2 w \in \underline{2}^{np}$. The condition $\epsilon_{u,v} = \epsilon_{\psi u, \psi v}$ means that we require $\epsilon \in C_{\text{cell}}^{\mathbb{Z}_p, 1}([0, 1]^{np}; \mathbb{F}_2)$. By Lemma 3.73, such ϵ exists.

Finally, we must also choose orderings of the circles at each resolution. In fact, any ordering of circles will do.

We must now describe the action of ψ on Kg . That is, forgetting the sign, we have ψ takes $Kg(u) \rightarrow Kg(\psi u)$ as in the proof of Proposition 3.72. Say $Z(\tilde{D}_u) = \{a_1, \dots, a_{\ell_1}\}$ so that $a_1 < \dots < a_{\ell_1}$ and $Z(\tilde{D}_{\psi u}) = \{b_1, \dots, b_{\ell_1}\}$ so that $b_1 < \dots < b_{\ell_1}$. For $x = a_1 \otimes \dots \otimes a_k \in Kg(u)$ taken to $b_{\sigma(1)} \otimes \dots \otimes b_{\sigma(k)} \in Kg(\psi u)$, the sign is just $\text{sgn}(\sigma)$.

We have now constructed ψ on objects of $\mathcal{AKHO}(\tilde{D})$ and $\mathcal{KHO}(\tilde{D})$. Since the edge assignment is equivariant, we have actions $\psi: \mathcal{AKHO}(\phi_{u,v}) \rightarrow \mathcal{AKHO}(\phi_{\psi u, \psi v})$ and $\psi: \mathcal{KHO}(\phi_{u,v}) \rightarrow \mathcal{KHO}(\phi_{\psi u, \psi v})$. The proof of the proposition now follows as in the proof of Proposition 3.72. To see nonsingularity of the resulting external action, consider any Khovanov generator $x \in Kg(u)$ fixed by ψ (viewed as a bijection, not a signed bijection). In particular, we have $\psi u = u$. Every invariant generator x either comes from a nontrivial circle of \tilde{D}_u , or is a product $x = x_{i_1} \dots x_{i_p}$ of trivial circles. In the former case, certainly ψ takes $x \rightarrow x$ with sign 1. In the latter, ψ acts by some permutation of $x_{i_1} \dots x_{i_p}$. To verify that the sign of ψ is 1, it suffices to check a particular ordering of the circles of u . To see this, note that reordering the circles changes the action of ψ (viewed as a permutation of $\{1, \dots, \ell\}$ using the ordering of the circles) by conjugation. Ordering

the trivial circles in a \mathbb{Z}_p -orbit by order of appearance, going counterclockwise starting from an arc $\tilde{\gamma}$, we see that, indeed, ψ acts with sign $+1$ on all fixed generators, for any ordering of circles. \square

3.6.3 Fixed-point functors

In this section, we find the fixed-point Burnside functors of the equivariant Khovanov-Burnside functors constructed above. The main result is the following.

Write $\iota: \underline{2}^n \hookrightarrow \underline{2}^{np}$ for the canonical embedding.

Theorem 3.75. *Let \tilde{D} be a p -periodic link diagram (with $p > 1$), with quotient diagram D . The Khovanov fixed-point functors are*

1. $\mathcal{AKH}(D) = \mathcal{KH}(\tilde{D})^{\mathbb{Z}_p}$
2. $\mathcal{AKH}^{j,k}(D) = \mathcal{AKH}^{pj-(p-1)k,k}(\tilde{D})^{\mathbb{Z}_p},$

for any pair of quantum and (k) -gradings (j, k) . If p is odd, we further have, for suitable choices of crossing orientations, edge assignments, and circle orderings at each resolution:

3. $\mathcal{AKHO}(D) = \mathcal{KHO}(\tilde{D})^{\mathbb{Z}_p},$
4. $\mathcal{AKHO}^{j,k}(D) = \mathcal{KHO}^{pj-(p-1)k,k}(\tilde{D})^{\mathbb{Z}_p}.$

Proof. Let us first address the case of $F = \mathcal{KH}(\tilde{D})$; that is, let us see that $F^{\mathbb{Z}_p} = \mathcal{AKH}(D)$. By Lemma 3.19, and the fact that the fixed-point category of $\underline{2}^{np}$ is the image of the canonical embedding $\underline{2}^n \rightarrow \underline{2}^{np}$, it suffices to identify $F^{\mathbb{Z}_p}(\iota u)$ for each $u \in \underline{2}^n$, as well as the correspondences $F^{\mathbb{Z}_p}(\phi_{\iota u, \iota v})$ for $u \geq_1 v$, and finally to identify the 2-morphisms associated to 2-dimensional faces of $\underline{2}^n$. Proposition 3.12 shows that $F^{\mathbb{Z}_p}(\iota u)$ is canonically identified with $\mathcal{AKH}(u)$. We package the proof that the 1-morphisms are

correct as Proposition 3.76 below, and the claim about 2-morphisms is Lemma 3.83. Assuming those lemmas, the present theorem follows directly. □

Proposition 3.76. *Let \tilde{D} be a p -periodic link diagram, with $p > 1$. Fix $u \geq_1 v \in \text{Ob}(\underline{2}^n)$ and consider a sequence of objects of $\underline{2}^{np}$ given by $\iota u \geq_1 u_1 \cdots \geq_1 u_p = \iota v$. Then*

$$\begin{aligned} \mathcal{KH}(\tilde{D})^{\mathbb{Z}_p}(\phi_{u_{p-1}, \iota v} \circ \cdots \circ \phi_{\iota u, u_1}) &\cong \mathcal{AKH}(D)(\phi_{u, v}), \\ \mathcal{AKH}(\tilde{D})^{\mathbb{Z}_p}(\phi_{u_{p-1}, \iota v} \circ \cdots \circ \phi_{\iota u, u_1}) &\cong \mathcal{AKH}(D)(\phi_{u, v}), \end{aligned} \tag{3.77}$$

where \cong denotes natural isomorphism. Further, if p is odd, then:

$$\begin{aligned} \mathcal{KHO}(\tilde{D})^{\mathbb{Z}_p}(\phi_{u_{p-1}, \iota v} \circ \cdots \circ \phi_{\iota u, u_1}) &\cong \mathcal{AKHO}(D)(\phi_{u, v}), \\ \mathcal{AKHO}(\tilde{D})^{\mathbb{Z}_p}(\phi_{u_{p-1}, \iota v} \circ \cdots \circ \phi_{\iota u, u_1}) &\cong \mathcal{AKHO}(D)(\phi_{u, v}), \end{aligned}$$

for appropriate choices for \tilde{D}, D of crossing orientations and edge assignments, and of circle orderings at each resolution.

Proof. First consider the case for $F = \mathcal{KH}(\tilde{D})$. By commutativity of the 2-dimensional faces of the cube, it suffices to show the identification of one-morphisms for any particular path $\{u_i\}_i$.

The proof amounts to a case-by-case check of the three different types of merges; see Figure 3.4.

First, say $\phi_{v, u}^{\text{op}}$ represents a $\mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{W}$ merge. Then $F(\iota u)$ has four invariant generators, $\{1, x_1, x_2, x_1 x_2\}$ where $x_1, x_2 \in Z(\tilde{D}_{\iota u})$, and $F(\iota v)$ has two invariant generators, $\{1, y_1 \cdots y_p\}$, for $y_1, \dots, y_p \in Z(\tilde{D}_{\iota v})$, all lying in the same \mathbb{Z}_p -orbit.

The first map $\phi_{\iota v, u_{p-1}}^{\text{op}}$ is a merge, and then all the following maps $\{\phi_{u_i, u_{i-1}}^{\text{op}}\}_{0 < i < p}$ are split maps. It is straightforward to check that $F(\phi_{\iota u, \iota v})^{\mathbb{Z}_p} \cong \{a_1, a_2\}$ with source and

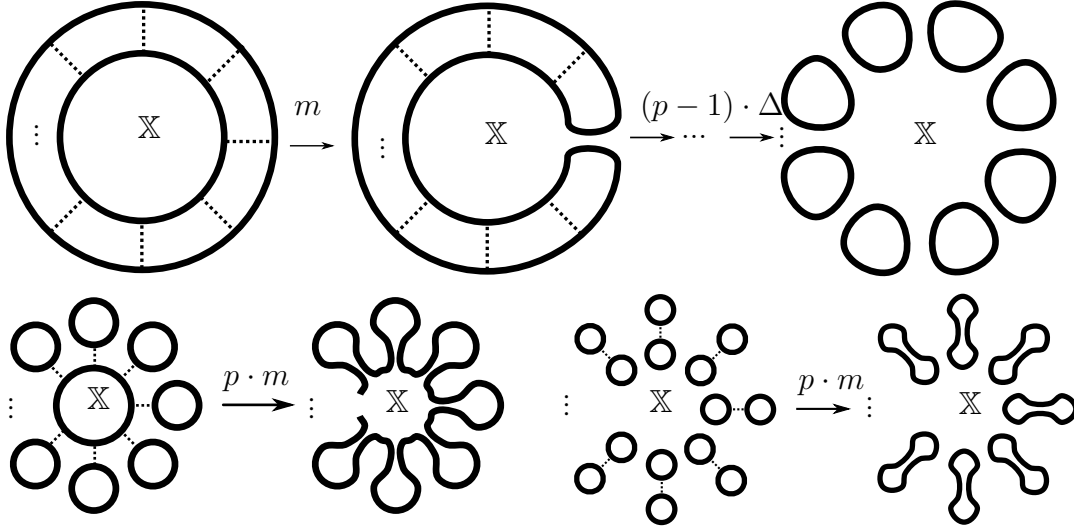


Figure 3.4: The three equivariant annular merges.

target maps $s(a_i) = y_1 \dots y_p$ and $t(a_i) = x_i$. Thus, $F(\phi_{\iota u, \iota v})^{\mathbb{Z}_p}$ is naturally isomorphic to $\mathcal{AKH}(D)(\phi_{u,v})$ for this case.

If $\phi_{v,u}^{\text{op}}$ represents a $\mathbb{V} \otimes \mathbb{W} \rightarrow \mathbb{V}$ merge, then all p maps $\{\phi_{u_i, u_{i-1}}^{\text{op}}\}_{1 \leq i \leq p}$ are merge maps. The invariant generators at ιv are $\{1, x, y, xy\}$ with $x \in Z(D_{\iota v})$ a nontrivial circle and $y \in Z(D_{\iota v})$ a trivial circle. The invariant generators at ιu are $\{1, z\}$ for $z \in Z(D_{\iota u})$. The correspondence $F(\phi_{\iota u, \iota v})^{\mathbb{Z}_p} = \{a_1, a_2\}$ with $s(a_1) = 1, s(a_2) = z$ and target $t(a_1) = 1, t(a_2) = x$. We then observe that $F(\phi_{\iota u, \iota v})^{\mathbb{Z}_p}$ is naturally isomorphic to $\mathcal{AKH}(D)(\phi_{u,v})$ in this case as well.

A similar situation occurs for the case $\mathbb{W} \otimes \mathbb{W} \rightarrow \mathbb{W}$. The invariant generators at \tilde{v} are $\{1, x_1 \dots x_p, y_1 \dots y_p, x_1 \dots x_p y_1 \dots y_p\}$, where $x_i \in Z(D_{\iota v})$ are all in the same \mathbb{Z}_p -orbit, and similarly for $y_i \in Z(D_{\iota v})$. The invariant generators at ιu are $\{1, z_1 \dots z_p\}$ where $z_i \in Z(D_{\iota u})$ lie in the same \mathbb{Z}_p -orbit. A quick check shows $F(\phi_{\iota u, \iota v})^{\mathbb{Z}_p} = \{a_1, a_2, a_3\}$ with $s(a_1) = 1, t(a_1) = 1$, and $s(a_2) = x_1 \dots x_p, t(a_2) = z_1 \dots z_p$, and finally $s(a_3) = y_1 \dots y_p, t(a_3) = z_1 \dots z_p$. It is then readily checked that $F(\phi_{\iota u, \iota v})^{\mathbb{Z}_p}$ is naturally isomorphic to $\mathcal{AKH}(D)(\phi_{u,v})$ in this case.

The above cases, along with duality, show that equation (3.77) holds.

The case of $F = \mathcal{AKH}(\tilde{D})$ is quite similar but somewhat simpler, and we omit it.

Next we treat the case $F = \mathcal{KH}\mathcal{O}(\tilde{D})$. We have already seen that, if we forget the signs, $(\mathcal{FF})^{\mathbb{Z}_p} = \mathcal{AKH}(D)$. Now, $\mathcal{AKH}\mathcal{O}(D)$ can be viewed as a way of sprinkling signs on the correspondences of $\mathcal{AKH}(D)$ (and similarly for $\mathcal{KH}\mathcal{O}(\tilde{D})$ relative to $\mathcal{KH}(\tilde{D})$), and we need to say that these sprinklings respect the equality of Burnside functors in (3.77).

Recall that in order to define F , we needed to choose the data of an (equivariant) orientation of crossings, as well as an equivariant edge assignment. Say we have fixed these data. Now, in order to define $\mathcal{AKH}\mathcal{O}(D)$, we need an orientation of crossings of D , as well as an edge assignment of D . We choose the orientation of crossings coming from taking the quotient of the orientation of crossings of \tilde{D} . In order to compare $\mathcal{AKH}\mathcal{O}(D)$ with F , we must find a way to define an edge assignment on D , given the edge assignment upstairs. We start with the following lemma. Recall that $Kg(\tilde{D})^{\mathbb{Z}_p}$ upstairs is identified with $Kg(D)$ downstairs, using the choice of an arc $\tilde{\gamma}$, as in the discussion after Proposition 3.12.

Lemma 3.78. *Let C be an index-1 annular resolution configuration, with associated odd annular Khovanov projective functor $\mathfrak{F}'_{\text{Ann}_o} : \underline{2}^{\text{op}} \rightarrow \mathbb{Z}\text{-Mod}$. Let p odd, and let \tilde{C} denote the p -cover of C , with some choice of lift of γ to $\tilde{\gamma}$. Set $v_i = 0^{p-i}1^i \in \text{Ob}(\underline{2}^p)$. Let $\mathfrak{F}'_o : (\underline{2}^p)^{\text{op}} \rightarrow \mathbb{Z}\text{-Mod}$ denote the odd Khovanov projective functor associated to \tilde{C} . Then*

$$(\mathfrak{F}'_o(\phi_{v_{p-1}, v_p}^{\text{op}}) \circ \cdots \circ \mathfrak{F}'_o(\phi_{v_0, v_1}^{\text{op}}))_{\mathbb{Z}_p} = \mathfrak{F}'_{\text{Ann}_o}(\phi_{0,1}^{\text{op}}). \quad (3.79)$$

Here we have written $(\cdot)_{\mathbb{Z}_p}$ to denote the restriction of (\cdot) to $\mathfrak{F}'_o(0^p)^{\mathbb{Z}_p}$ and then its projection to $\mathfrak{F}'_o(1^p)^{\mathbb{Z}_p}$. Further, recall that the ordering of the arcs and circles of \tilde{C} are defined with respect to the lift $\tilde{\gamma}$.

Proof. The proof is a case-by-case check of index-1 annular resolution configurations.

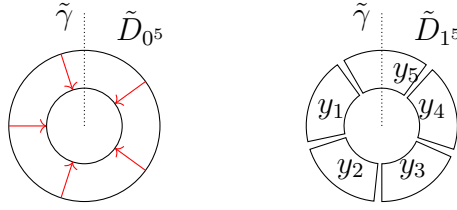
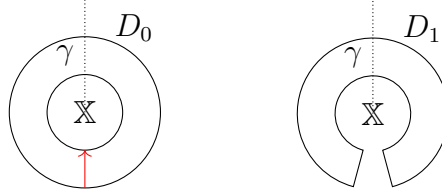


Figure 3.5: **The $\mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{W}$ case for $p = 5$.** The map downstairs is $x_1, x_2 \rightarrow x$, where $x_1, x_2 \in Z(D_0)$, $x \in Z(D_1)$. Let $Z(\tilde{D}_{0^5}) = \{\tilde{x}_1, \tilde{x}_2\}$, the elements over x_1, x_2 , and let $Z(\tilde{D}_{1^5}) = \{y_1, \dots, y_5\}$, related by the action of \mathbb{Z}_5 . Then upstairs the map on fixed points is $\tilde{x}_i \rightarrow (y_5 - y_1)(y_1 - y_2)(y_2 - y_3)(y_3 - y_4)y_5 = y_1y_2y_3y_4y_5$, which verifies Lemma 3.78 in this example.

First, consider the resolution configuration associated to a merge $\mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{W}$. That is, say we have the following picture in the base:



For this case, consider Figure 3.5, which illustrates the proof for $p = 5$; the proof for general p is entirely analogous.

Next, consider the resolution configuration associated to a merge $\mathbb{V} \otimes \mathbb{W} \rightarrow \mathbb{V}$. In this case, both upstairs and downstairs there are only merge maps, from which the Lemma follows readily.

Next, consider the case $\mathbb{W} \otimes \mathbb{W} \rightarrow \mathbb{W}$. In this case, again, upstairs there are only merges, from which the result is immediate.

Next, consider the case $\mathbb{W} \rightarrow \mathbb{W} \otimes \mathbb{W}$. For this, downstairs we have $Z(D_0) = \{x\}$, $Z(D_1) = \{y_1, y_2\}$ and upstairs $Kg(0^p)^{\mathbb{Z}_p} = \{z_1 \dots z_p\}$ and $Kg(1^p)^{\mathbb{Z}_p} = \{w_1^1 \dots w_p^1, w_1^2 \dots w_p^2, w_1^1 \dots w_p^1 w_1^2 \dots\}$ where the ordering is so that $y_1 < y_2$ and $w_1^1 \leq w_i^1 < w_1^2 \leq w_i^2$, and $\{z_i\}, \{w_i^1\}, \{w_i^2\}$ are the orbits of circles z_1, w_1^1, w_1^2 , respectively, under \mathbb{Z}_p .

Downstairs, having fixed an orientation of crossing going from y_1 to y_2 , we have:

$$\mathfrak{F}'_{\text{Ann}_o}(\phi_{0,1}^{\text{op}})(1) = y_1 - y_2 \quad \mathfrak{F}'_{\text{Ann}_o}(\phi_{0,1}^{\text{op}})(x) = y_1 y_2.$$

Upstairs, we observe, using the definition of the odd Khovanov projective functor:

$$\begin{aligned} \mathfrak{F}'_o(\phi_{v_{p-1}, v_p}^{\text{op}}) \circ \cdots \circ \mathfrak{F}'_o(\phi_{v_0, v_1}^{\text{op}})(1) &= (w_1^1 - w_1^2) \cdots (w_p^1 - w_p^2), \\ \mathfrak{F}'_o(\phi_{v_{p-1}, v_p}^{\text{op}}) \circ \cdots \circ \mathfrak{F}'_o(\phi_{v_0, v_1}^{\text{op}})(z_1 \cdots z_p) &= (w_1^1 - w_1^2) \cdots (w_p^1 - w_p^2)(z_1 \cdots z_p) \\ &= w_1^1 \cdots w_p^1 w_1^2 \cdots w_p^2. \end{aligned}$$

From this calculation, we have obtained the Lemma in the $\mathbb{W} \rightarrow \mathbb{W} \otimes \mathbb{W}$ case.

The cases $\mathbb{V} \rightarrow \mathbb{W} \otimes \mathbb{V}$ and $\mathbb{W} \rightarrow \mathbb{V} \otimes \mathbb{V}$ are very similar to the cases we have done so far, and we leave them as exercises to the reader; this finishes the proof of Lemma 3.78.

□

Now, we must see how to go from an (equivariant, type X) edge assignment $\tilde{\epsilon}$ on \tilde{D} to an edge assignment on D . Fix $u \geq_1 v \in \underline{2}^n$. We define $v_i \in \underline{2}^{np}$ by $v_i = (v)^{p-i}(u)^i$, as elements of $(\underline{2}^n)^p$ for $0 \leq i \leq p$. We then define an element $\epsilon \in C_{\text{cell}}^1([0, 1]^n; \mathbb{Z}_2)$ by:

$$\epsilon_{u,v} = \tilde{\epsilon}_{v_p, v_{p-1}} \cdots \tilde{\epsilon}_{v_1, v_0}.$$

Recall the definition of the *obstruction cocycle* $\Omega(D)$ from Section 3.2.4. Any cochain $c \in C_{\text{cell}}^1([0, 1]^n; \mathbb{Z}_2)$ for which $\delta c = \Omega(D)$ gives a functor $\mathcal{AKHO}(D)_c: \underline{2}^n \rightarrow \mathcal{B}_{\mathbb{Z}_2}$, the odd annular Khovanov-Burnside functor with edge assignment ϵ , whose stable equivalence class is well-defined, i.e. independent of c . To proceed, we need to confirm that $\delta \epsilon = \Omega(D)$. We will work with the type X obstruction cocycle; the following lemma also holds for the type Y obstruction cocycle, if the edge assignment upstairs is chosen to be type Y (the proof below immediately generalizes to the type Y case).

Lemma 3.80. For $\epsilon \in C_{\text{cell}}^1([0, 1]^n; \mathbb{Z}_2)$ as defined above, we have $\delta\epsilon = \Omega(D)$.

Proof. For $x \in C_{\text{cell}}^2([0, 1]^n; \mathbb{Z}_2)$ and $u \geq_2 w \in \underline{2}^n$, we write $x_{u,w}$ for the evaluation of x on the copy of $[0, 1]^2$ corresponding to the pair (u, w) . We need to check that for each 2-dimensional face $u \geq_1 v, v' \geq_1 w$, that $(\delta\epsilon)_{u,w} = \Omega(D)_{u,w}$. There are two cases to consider.

First, say that $\mathfrak{F}'_{\text{Ann}_o}(\phi_{v,u}^{\text{op}})\mathfrak{F}'_{\text{Ann}_o}(\phi_{w,v}^{\text{op}}) \neq 0$. Then $\Omega(D)_{u,w}$ is determined as follows:

$$\mathfrak{F}'_{\text{Ann}_o}(\phi_{v,u}^{\text{op}})\mathfrak{F}'_{\text{Ann}_o}(\phi_{w,v}^{\text{op}}) = \mathfrak{F}'_{\text{Ann}_o}(\phi_{v',u}^{\text{op}})\mathfrak{F}'_{\text{Ann}_o}(\phi_{w,v'}^{\text{op}})$$

if and only if $\Omega(D)_{u,w} = 1$. However, if $\mathfrak{F}'_{\text{Ann}_o}(\phi_{v,u}^{\text{op}})\mathfrak{F}'_{\text{Ann}_o}(\phi_{w,v}^{\text{op}}) = 0$, more data is needed to determine $\Omega(D)_{u,w}$. For comparison, if we worked with \mathfrak{F}'_o in place of $\mathfrak{F}'_{\text{Ann}_o}$, more data is needed to define $\Omega(D)_{u,w}$ only for ladybug resolution configurations $C_{u,w}$ (in that case $\Omega(D)_{u,w} = -1$ for type X edge assignments, etc.).

Let us consider the case where $\mathfrak{F}'_{\text{Ann}_o}(\phi_{v,u}^{\text{op}})\mathfrak{F}'_{\text{Ann}_o}(\phi_{w,v}^{\text{op}}) \neq 0$. Write $w_i = w^{p-i}v^i$, $w'_i = w^{p-i}v'^i$ and $v_i = v^{p-i}u^i$, $v'_i = v'^{p-i}u^i$, as objects in $\underline{2}^{np}$. Then

$$\begin{aligned} \tilde{\epsilon}_{v_p, v_{p-1}} \cdots \tilde{\epsilon}_{v_1, v_0} \tilde{\epsilon}_{w_p, w_{p-1}} \cdots \tilde{\epsilon}_{w_1, w_0} &= \tilde{\epsilon}_{v'_p, v'_{p-1}} \cdots \tilde{\epsilon}_{v'_1, v'_0} \tilde{\epsilon}_{w'_p, w'_{p-1}} \cdots \tilde{\epsilon}_{w'_1, w'_0} \\ \Leftrightarrow \quad \mathfrak{F}'_{\text{Ann}_o}(\phi_{v,u}^{\text{op}})\mathfrak{F}'_{\text{Ann}_o}(\phi_{w,v}^{\text{op}}) &= \mathfrak{F}'_{\text{Ann}_o}(\phi_{v',u}^{\text{op}})\mathfrak{F}'_{\text{Ann}_o}(\phi_{w,v'}^{\text{op}}), \end{aligned}$$

since

$$\begin{aligned} \tilde{\epsilon}_{v_p, v_{p-1}} \cdots \tilde{\epsilon}_{v_1, v_0} \tilde{\epsilon}_{w_p, w_{p-1}} \cdots \tilde{\epsilon}_{w_1, w_0} \mathfrak{F}'_o(\phi_{v_{p-1}, v_p}^{\text{op}}) \circ \cdots \circ \mathfrak{F}'_o(\phi_{v_0, v_1}^{\text{op}}) \mathfrak{F}'_o(\phi_{w_{p-1}, w_p}^{\text{op}}) \circ \cdots \circ \mathfrak{F}'_o(\phi_{w_0, w_1}^{\text{op}}) = \\ \tilde{\epsilon}_{v'_p, v'_{p-1}} \cdots \tilde{\epsilon}_{v'_1, v'_0} \tilde{\epsilon}_{w'_p, w'_{p-1}} \cdots \tilde{\epsilon}_{w'_1, w'_0} \mathfrak{F}'_o(\phi_{v'_{p-1}, v'_p}^{\text{op}}) \circ \cdots \circ \mathfrak{F}'_o(\phi_{v'_0, v'_1}^{\text{op}}) \mathfrak{F}'_o(\phi_{w'_{p-1}, w'_p}^{\text{op}}) \circ \cdots \circ \mathfrak{F}'_o(\phi_{w'_0, w'_1}^{\text{op}}). \end{aligned}$$

That is, we have verified $(\delta\epsilon)_{u,w} = \Omega(D)_{u,w}$ on all faces of the first case.

We next treat faces of the second type. We start by cataloging such faces:

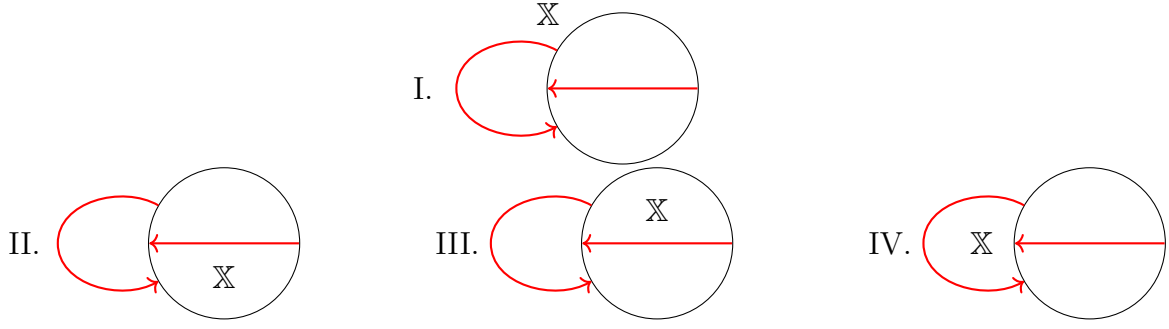


Figure 3.6: The annular resolution configurations of type X.

Lemma 3.81. *Say $u \geq_1 v \geq_1 w$ and let $C_{u,w}$ be an index-2 odd annular resolution configuration so that*

$$\mathfrak{F}'_{\text{Ann}_o}(\phi_{v,u}^{\text{op}})\mathfrak{F}'_{\text{Ann}_o}(\phi_{w,v}^{\text{op}}) = 0. \quad (3.82)$$

Then either the underlying resolution configuration of $C_{u,w}$ is type X or Y, or $C_{u,w}$ consists of three concentric nontrivial circles C_1, C_2, C_3 with C_1, C_2 joined by an arc, as well as C_2, C_3 joined by an arc, or the dual configuration of the latter.

Proof. The proof of this Lemma is a simple case-by-case check. □

Next, we check that $(\delta\epsilon)_{u,w} = \Omega(D)_{u,w}$ for configurations $C_{u,w}$ of type X or Y. We may as well assume now that $u = 11, v = 10, v' = 01, w = 00$, to simplify notation. First consider $C_{u,w}$ of type X. There are four annular resolution configurations to consider, pictured in Figure 3.6.

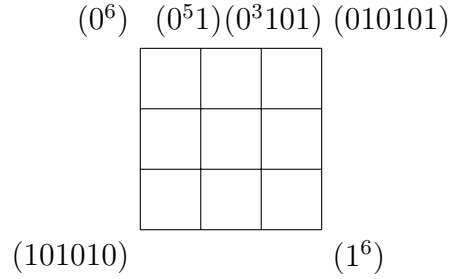
Recall that we need to show

$$\tilde{\epsilon}_{v_p, v_{p-1}} \cdots \tilde{\epsilon}_{v_1, v_0} \tilde{\epsilon}_{w_p, w_{p-1}} \cdots \tilde{\epsilon}_{w_1, w_0} = -\tilde{\epsilon}_{v'_p, v'_{p-1}} \cdots \tilde{\epsilon}_{v'_1, v'_0} \tilde{\epsilon}_{w'_p, w'_{p-1}} \cdots \tilde{\epsilon}_{w'_1, w'_0}.$$

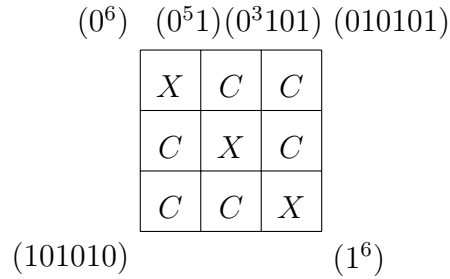
However, we have, by definition of an edge assignment,

$$\prod_{i=1}^p \tilde{\epsilon}_{v_i, v_{i-1}} \tilde{\epsilon}_{v'_i, v'_{i-1}} \tilde{\epsilon}_{w_i, w_{i-1}} \tilde{\epsilon}_{w'_i, w'_{i-1}} = \prod_{(a,c) \in \mathbb{I}} \Omega(\tilde{D})_{a,c},$$

where \mathbb{I} is the set of pairs (a, c) with $c = 0^2x \in (\underline{2}^2)^p$ for some $x \in (\underline{2}^2)^{p-1}$, and a is the result of replacing the rightmost 0 in the first $\underline{2}^p$ -factor of c with a 1, and the rightmost 0 in the second $\underline{2}^p$ -factor with a 1; e.g. $(01^5, 0^2101^2) \in \mathbb{I}$ for $p = 3$. We draw the product $\prod \Omega(\tilde{D})$ as a product running over the faces of a grid, whose vertices are objects of $(\underline{2}^2)^p$. We visualize this as follows in the $p = 3$ case, with only a few vertices labeled:



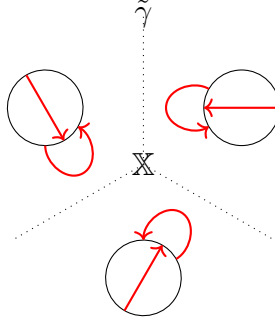
Each of the faces of this grid G , corresponding to $a \geq_1 b, b' \geq_1 c \in (\underline{2}^2)^p$, is assigned a label in $\{A, C, X, Y\}$ according to the type of the corresponding odd resolution configuration $\tilde{D}_{a,c}$. Sometimes, we will assign the faces of the grid a ± 1 , using that $\Omega(\tilde{D})_{a,c} = 1$ for faces of type C,Y and is -1 for faces of type A,X. We will work to understand this grid in cases I-IV. For instance, we will see below that for case I and $p = 3$, the grid is:



Given a vertex $c \in \text{Vert}(G)$, with vertex $b \in \text{Vert}(G)$ directly below, and $b' \in \text{Vert}(G)$ directly to the right, we call D_b the *left resolution* of D_c , and $D_{b'}$ the *right resolution* of D_c . Note that each edge of the grid corresponds to resolving a crossing that is entirely contained within a single sector (recalling the notation of sectors from Section 3.2.6),

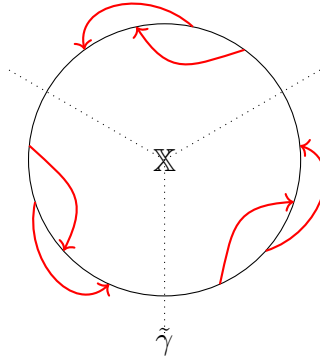
and so we may label each edge of the grid by the sector in which the corresponding surgery occurs.

First we treat the configuration I. Here, upstairs we have a picture as:



which illustrates the $p = 3$ case. Let G denote the grid associated to such a configuration. It is immediate from the definitions that all the faces on the main diagonal of G are type X. Now, for each off-diagonal face D , we see that one of the resolutions performed must be a merge. Moreover, each off-diagonal resolution configuration is disconnected. Inspecting the list of odd 2-dimensional resolution configurations, any such configuration is of type C, and so we have verified Lemma 3.80 in this case.

Next, we treat case II. The picture upstairs is as follows, again illustrated for $p = 3$:



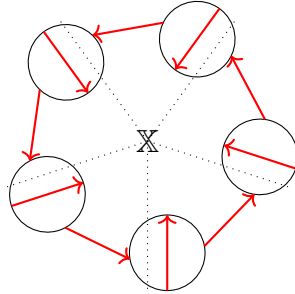
It is readily checked once again that all of the diagonal faces are type X. Fix an off-diagonal face with upper-left hand vertex at $a \in \text{Vert}(G)$, whose left-resolution is in the

q^{th} sector and whose right-resolution is in the $r \neq q^{\text{th}}$ sector. Write $\underline{2}_t^2$ for the t^{th} -factor of $\underline{2}^2$ in $(\underline{2}^2)^p$. Then the resulting resolution configuration depends only on the initial condition of c in $\underline{2}_r^2$ and $\underline{2}_q^2$. To see this, consider the restriction of $D_{a,c}$ to a sector S_t outside of S_q and S_r . It will be an arc connecting the boundary components $\partial^+ S_t$ and $\partial^- S_t$ (where the positive (negative) boundary $\partial^+ S_t$ ($\partial^- S_t$) of a sector S_t will denote the end obtained by traversing counterclockwise (clockwise)), as well as some disjoint circles, no matter the restriction of c to $\underline{2}_t^2$. In particular, the resulting two-dimensional resolution configuration $D_{a,c}$ is formed by drawing the parts of the resolution configuration in the q and r sectors, and attaching these on their boundaries; see for example Figure 3.7.

Next, fix $c \in \text{Vert}(G)$, the upper-left hand corner of a square a, b, b', c in G , where D_b is the left resolution and $D_{b'}$ is the right resolution. Say the pair $a \geq_2 c$ differs only in entries e_1, e_2 , where e_1 is in the q^{th} -sector and e_2 is in the r^{th} -sector. Let a_q, a_r, c_q, c_r denote the restrictions of a and c to $\underline{2}_q^2, \underline{2}_r^2$, respectively, and recall that the type of the resolution configuration $D_{a,c}$ depends only on a_q, c_q, a_r, c_r . Note furthermore that the only c in the grid for which $c_q = c_r = 0^2$ is $c = 0^{2p}$, which does not participate in an off-diagonal face. So, we need only consider pairs (a, c) with $(c_q, c_r) \neq (0^2, 0^2)$. We list all such resolution configurations and their types in Figure 3.7. Indeed, we see that all the off-diagonal faces of G are type C, which completes case II (since type X faces appear an odd number of types on the diagonal).

Case III is quite similar to case II and we leave it to the reader.

Finally, we address case IV. The picture upstairs is as follows (for $p = 5$):



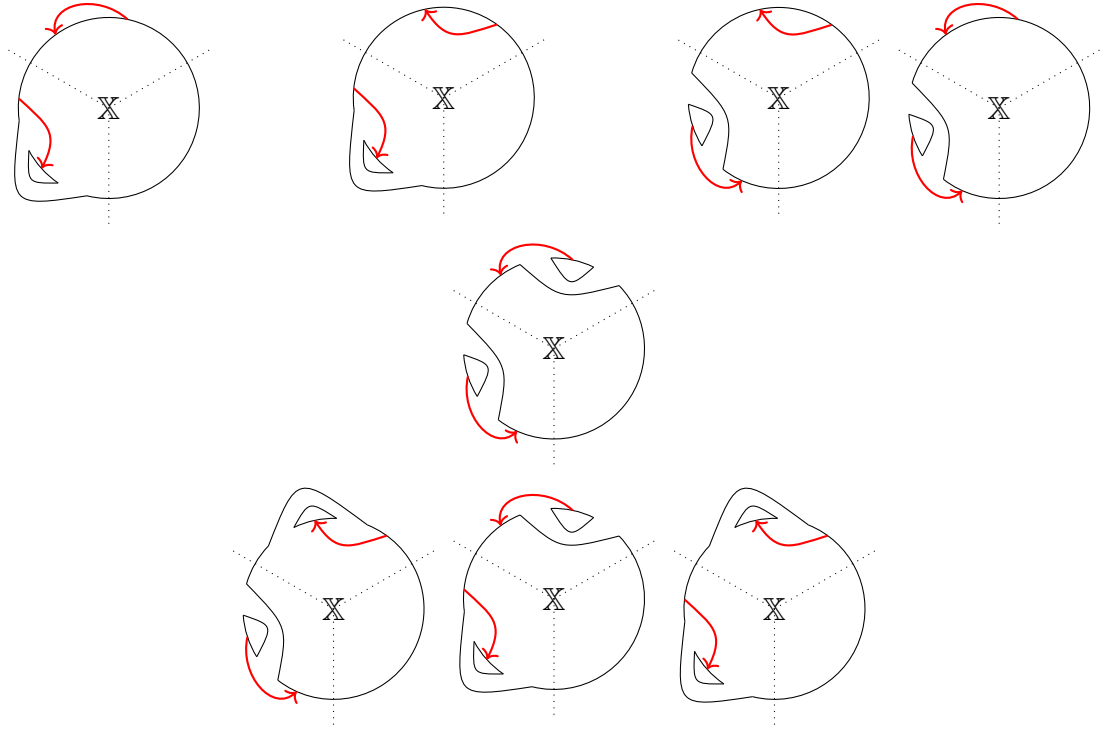


Figure 3.7: The off-diagonal resolution configurations in case II. The first four configurations are realized up to isotopy by expressions of the form $(*1, 0*) \in \underline{\mathbb{Z}}_q^2 \times \underline{\mathbb{Z}}_r^2$ and their permutations, while the latter four are obtained from permutations of $(*1, *1) \in \underline{\mathbb{Z}}_q^2 \times \underline{\mathbb{Z}}_r^2$.

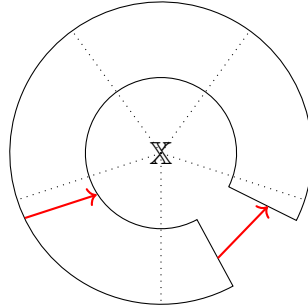
We order the crossings so that the edges forming a pentagon correspond to the first factor $\underline{2}^p \rightarrow (\underline{2}^2)^p$ and the other edges correspond to the second factor $\underline{2}^p \rightarrow (\underline{2}^2)^p$.

We divide length 1-arrows in $(\underline{2}^2)^p$ into two sets as follows. Recall that each arrow $\phi_{v,u}^{\text{op}}$ for $u \geq_1 v$ can be recorded as the element $v \in (\underline{2}^2)^p$, but with one of the 1, 0-entries of v replaced by a $*$ to denote the entry that changes between v, u . If $*$ is at an odd position in $\underline{2}^p$ (that is, $*$ occurs in the first $\underline{2}$ -factor in some copy $\underline{2}^2 \subset (\underline{2}^2)^p$), we call $\phi_{v,u}^{\text{op}}$ a *left edge*, otherwise a *right edge*. Similarly, an index-2 resolution configuration from $u \geq_2 w$ can be described by an element in $(\underline{2}^2)^p$ with two bits replaced by $*$.

Note that resolving a right edge on some resolution D_c is a split, unless $c = (10)^p$. Further, resolving a left edge is a merge unless $c = (10)^k(00)(10)^{p-k-1}$ for some k .

Further, any resolution configuration $D_{u,w}$ for which $\phi_{w,v}^{\text{op}}$ is a split and $\phi_{v',u}^{\text{op}}$ is a split, while $\phi_{w,v'}^{\text{op}}$ and $\phi_{v,u}^{\text{op}}$ are merges, has type C.

Then we need only consider faces in G containing the vertex $(10)^p$ or some $(10)^k(00)(10)^{p-k-1}$. However, $(10)^k(00)(10)^{p-k-1}$ is a vertex of G if and only if $k = 0$. So, we see in fact that only the lower left-hand cornered can be of type other than C. The picture in the lower left-hand corner is:

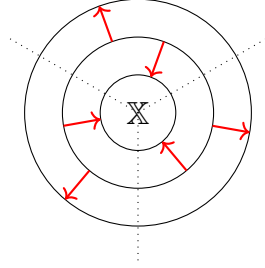


This is a type X face, and so the proof is completed for case IV.

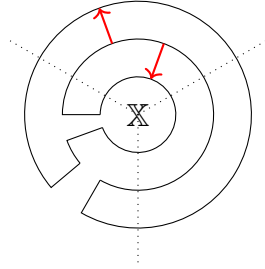
Translating the above proof to type Y faces is immediate. The grid is the same in each case, with type X faces replaced with type Y faces.

The only case that remains to check is that of three concentric circles (and its dual). We fix an orientation of edges as below; the case of other orientations is safely left to

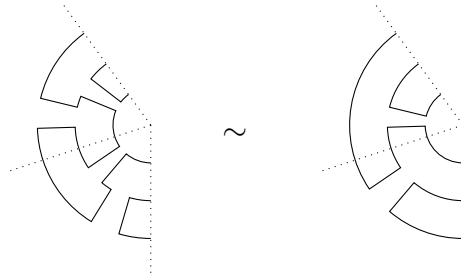
the reader.



We order the crossings so that the outer edges correspond to the first factor $\underline{2}^p \rightarrow (\underline{2}^2)^p$ and the inner edges correspond to the second factor $\underline{2}^p \rightarrow (\underline{2}^2)^p$. The upper left-hand corner of G is readily seen to be a type C configuration, since it consists of two merges. We note that the next configuration on the diagonal of G is a type X face:



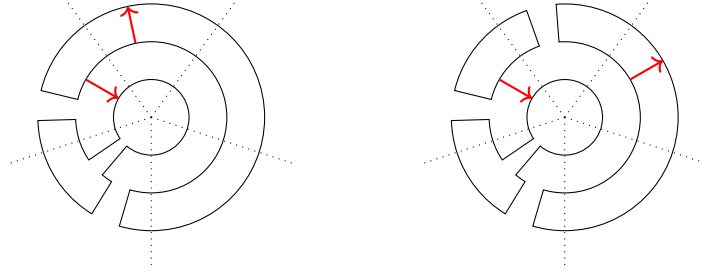
In fact, all other faces on the diagonal are type X, since the arcs outside of the ‘active’ sector, up to isotopy, do not depend on c , as is illustrated below:



In particular, there are an even number of faces of type X on the diagonal.

For $u \in (\underline{2}^2)^p$, let $|u|_1$ (resp. $|u|_2$) denote the number of 1’s occurring in the first copy of $\underline{2}^p \rightarrow (\underline{2}^2)^p$ (resp. second copy). Now, say $D_{a,c}$ is an index-2 resolution configuration

so that $|c|_1 > |c|_2$, for $a, c \in \text{Vert}(G)$, and b is the left resolution, b' the right resolution. Such resolution configurations are, up to isotopy:



From these, we observe that $\phi_{c,b'}^{\text{op}}$ is a merge and $\phi_{b',a}^{\text{op}}$ is a split, while $\phi_{c,b}^{\text{op}}$ is a split and $\phi_{b,a}^{\text{op}}$ is a merge. Any such resolution configuration has type C. For any c with $|c|_2 > |c|_1$, it turns out similarly that $D_{a,c}$ is type C. Then, for the case of three concentric circles downstairs, $(\delta\epsilon)_{u,w} = 1 = \Omega(D)_{u,w}$. (The case of three concentric circles, with the orientation of edges changed, results in replacing the type X faces on the diagonal by type Y faces).

We omit the case dual to three concentric circles; it follows by application of techniques similar to above.

Since there is at most one signed matching compatible with the Khovanov-Burnside functor, we have that the matching specified above is the ladybug matching. This completes the proof of Lemma 3.80. \square

In turn, Lemma 3.78 and Lemma 3.80 complete the proof Proposition 3.76. \square

We next deal with the case of 2-morphisms for the even Khovanov functor. Note that, for p odd, the following lemma is a consequence of Lemma 3.80.

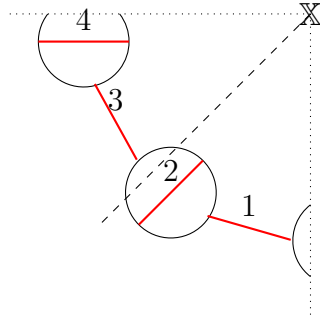
Lemma 3.83. *Let $u \geq_1 v, v' \geq_1 w \in \underline{2}^n$. The bijection $\mathcal{KH}(\phi_{\iota v, \iota w})^{\mathbb{Z}_p} \circ \mathcal{KH}(\phi_{\iota u, \iota v})^{\mathbb{Z}_p} \rightarrow \mathcal{KH}(\phi_{\iota v', \iota w})^{\mathbb{Z}_p} \circ \mathcal{KH}(\phi_{\iota u, \iota v'})^{\mathbb{Z}_p}$ is the ladybug matching.*

Proof. This is quite similar to, but easier than, the proof of Lemma 3.80 above. First of all, there is only something to check if the configuration $D_{u,w}$ downstairs is a ladybug (so

there is no analogue of the three-concentric circles case in the previous proof). Moreover, $\mathcal{KH}(\phi_{\iota v, \iota w})^{\mathbb{Z}_p} \circ \mathcal{KH}(\phi_{\iota u, \iota v})^{\mathbb{Z}_p} = \emptyset$ for configurations of type II and III (appearing in the proof of Lemma 3.80). That is, we need only consider index 2 annular resolution configurations downstairs of types I and IV.

With the experience from Lemma 3.80, the case I is rather direct, and is safely left to the reader. For case IV, we argue inductively. For odd p , we are already done, and the reader may readily verify that the Lemma holds for the case $p = 2$.

Say we have verified case IV for fixed p' , we show how to verify it for $p = 2p'$. The resolution configuration \tilde{D} upstairs is formed from p' sectors of the form:



We now draw the grid G as in the odd case, except that we order the crossings using the ordering of $(\mathbb{Z}^4)^{p'}$, rather than $(\mathbb{Z}^2)^p$. That just means that in the above picture, we resolve all edges labeled ‘1’ (resp. 2) before any of those labeled ‘3’ (4).

The $\mathbb{Z}_{p'}$ -fixed resolutions look as in Figure 3.8, in one of the p' sectors.

In the configuration \tilde{D}_{1010} , label the inner circle by x and the outer circle by y .

Using our inductive hypothesis (and looking at the ladybug matching on $\tilde{D}/\mathbb{Z}_{p'}$), the circle x is matched with $z_1 \dots z_{p'}$, where z_i are the circles in \tilde{D}_{1100} that intersect sector boundaries. A further use of our inductive hypothesis matches $z_1 \dots z_{p'}$ with the product $w_1^1 \dots w_{p'}^1 w_1^2 \dots w_{p'}^2$, where the w^1, w^2 are as labeled in Figure 3.8. Note that the generator $w_1^1 \dots w_{p'}^2$ is indeed \mathbb{Z}_p -invariant, as are x and y . Taking the quotients of \tilde{D}_{1010} and \tilde{D}_{0101} by \mathbb{Z}_p , we see that \bar{x} , the generator downstairs corresponding to x , indeed corresponds,

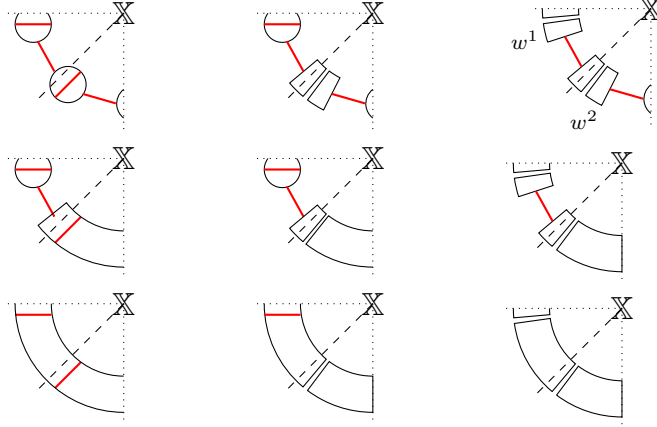


Figure 3.8: **Resolution configurations appearing in Lemma 3.83.** Here are pictured (one sector of) the resolution configurations invariant under the action of $\mathbb{Z}_{p'} \subset \mathbb{Z}_p$. The configurations on the top row are \tilde{D}_{0^4} , \tilde{D}_{0100} , \tilde{D}_{0101} , followed by the row \tilde{D}_{1000} , \tilde{D}_{1100} , \tilde{D}_{1101} and finally \tilde{D}_{1010} , \tilde{D}_{1110} , \tilde{D}_{1^4} . We have simplified the indexing by writing the indices for the quotient diagram $\tilde{D}/\mathbb{Z}_{p'}$.

under the right ladybug matching, to \bar{w} , the generator downstairs corresponding to the product $w_1^1 \dots w_{p'}^2$. This establishes the inductive step, and completes the Lemma.

This lemma may also be proved more directly, and indeed it is a pleasant exercise to verify the odd case without taking advantage of Lemma 3.80.

□

3.6.4 Well-definedness of the action

In this section we show that, for a p -periodic link \tilde{L} , the \mathbb{Z}_p -external stable equivalence class of the Burnside functor \mathcal{KH} is an invariant of \tilde{L} , and moreover if p is odd that the external equivariant stable equivalence class of $\mathcal{KH}\mathcal{O}$ is an invariant of \tilde{L} , and corresponding statements for the annular functors \mathcal{AKH} and $\mathcal{AKH}\mathcal{O}$.

Proof of Theorem 3.71. Throughout the proof we will usually abbreviate ‘(equivariant) external stable equivalence class’ to ‘equivalence class,’ where it will cause no confusion. We start with the case of p odd and $\mathcal{KH}\mathcal{O}$. We must first show that the equivalence class of $\mathcal{KH}\mathcal{O}(\tilde{D})$, for a fixed diagram \tilde{D} , is an invariant of the choices made

in its construction. Namely, we show independence of the orientation of crossings, the (equivariant) edge assignment, and the ordering of the circles a_i at each resolution. The proof of these claims almost follows verbatim from the start of the proof of Theorem 1.7 of [SSS18].

- Edge assignment:** Let ϵ, ϵ' be two different equivariant edge assignments of the same type. As noted in [ORSz13, Lemma 2.2], $\epsilon\epsilon'$ is a (multiplicative) (\mathbb{Z}_p -invariant) cochain in $C_{\text{cell}}^1([0, 1]^n; \mathbb{Z}_2)$. By Lemma 3.73, $\epsilon\epsilon'$ is the coboundary of an invariant 0-cochain α on the cube of resolutions. That is, there is a map $\alpha: \underline{2}^n \rightarrow \{\pm 1\}$, so that for any $v \geq_1 w$ $\alpha(v)\alpha(w) = \epsilon(\phi_{w,v}^{\text{op}})\epsilon'(\phi_{w,v}^{\text{op}})$. If F_0 and F_1 are the corresponding functors $\underline{2}^n \rightarrow \mathcal{B}_K$, we construct a stable equivalence using the functor $F_2: \underline{2}^{n+1} \rightarrow \mathcal{B}_{\mathbb{Z}_2}$, defined by $F_2|_{i \times \underline{2}^n} = F_i$, and on the arrows between the two copies of $\underline{2}^n$ using the signed (identity) correspondence $F_1(v) \rightarrow F_2(v)$ determined by α . That is, we apply the sign reassignment by α in the language of [SSS18, Definition 3.5]. Using the invariance of α , we see that F_2 admits an external action. It is straightforward that this natural transformation induces quasi-isomorphisms on the totalization of all fixed-point functors, finishing this check.
- (Equivariant) Orientations at crossings:** Recall that [ORSz13, Lemma 2.3] asserts that for oriented diagrams (L, o) and (L, o') and an edge assignment ϵ for (L, o) , there exists an edge assignment of the same type ϵ' for (L, o') so that $Kc_o(L, o, \epsilon) \cong Kc_o(L, o', \epsilon')$. The isomorphism constructed in that Lemma respects the Khovanov generators, and so induces an isomorphism of Burnside functors. Some thought shows that the natural generalization to the equivariant setting also holds; that is, for a change of equivariant orientation of crossing, the corresponding odd Khovanov chain complexes are identified (and ϵ' is equivariant), from which independence of \mathcal{KHO} follows. (Independence of the (equivariant) orientations of crossings can also be proved using (equivariant) Reidemeister II moves twice, as in [SSSz17, Figure 4.5].)

- **Type of edge assignment:** [ORSz13, Lemma 2.4] shows that an edge assignment ϵ of a link diagram with oriented crossings (L, o) of type X can also be viewed as a type Y edge assignment for some orientation o' . That is, the type X Burnside functor associated to (L, o, ϵ) is already the type Y Burnside functor associated to (L, o', ϵ) . In fact, if L is a periodic link diagram, the orientation o' constructed in [ORSz13] is equivariant. Moreover, the identification of the Burnside functors is equivariant, handling this case.
- **Ordering of circles at each resolution:** We must check that reordering the circles of a resolution results in an equivalent Burnside functor. For this, let $Kg(u)$ and $Kg'(u)$ denote the Khovanov generators for two differing orderings of the circles for a fixed link diagram. These orderings are related by a bijection from $Kg(u)$ to $Kg'(u)$. One checks directly that these bijections relate the two functors $F_1, F_2: \underline{2}^n \rightarrow \mathcal{B}_K$ by a sign reassignment, which, moreover, *commutes with the action of \mathbb{Z}_p* .

We now assume that the ordering of the circles upstairs is chosen as at the end of Section 3.2.6.

We show how to check invariance of \mathcal{KHO} under Reidemeister moves by upgrading the proof for chain complexes to Burnside functors, as is done in [LLS17a],[LS14], with the only change that we keep track of the external action in the course of the proof. We will work out the details in the case of a Reidemeister I move - this case will make clear what modifications are necessary to the usual invariance proof of \mathcal{KHO} (without external action) for Reidemeister II and III moves. Indeed, the proof of invariance is largely an iterated version of the usual invariance proof of Khovanov homology.

Let \tilde{D} be a periodic link diagram, and let \tilde{D}' be a diagram that differs from \tilde{D} by only an equivariant Reidemeister 1 (R1) move, which consists of p usual Reidemeister moves in the same orbit. See Figure 3.9, where we choose one of the R1 moves for concreteness. Let F_1 denote the odd Khovanov-Burnside functor of \tilde{D} , and F_2 that of \tilde{D}' .

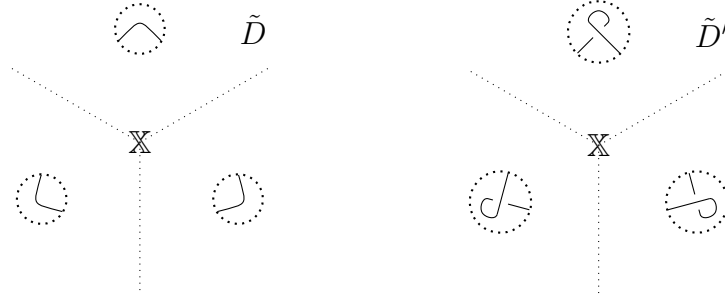


Figure 3.9: An equivariant Reidemeister I move. The left-hand image denotes a periodic link diagram \tilde{D} (with $p = 3$ pictured), with a \mathbb{Z}_p -orbit of a certain unknotted arc in picked out. The right-hand image denotes the periodic link diagram \tilde{D}' obtained by performing a Reidemeister I move along each arc of the orbit.

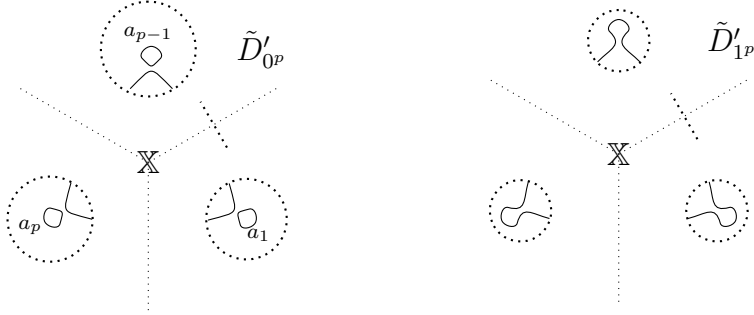


Figure 3.10: Some resolutions of the link diagram \tilde{D} . The ellipses to the upper-right record that we have omitted all but three sectors of the periodic link diagram \tilde{D}' .

From its definition $Kg(\tilde{D}') = \coprod_{i \in \underline{2}^p} Kg(\tilde{D}'_i)$, where \tilde{D}'_i denotes the resolution of \tilde{D}' by resolving the orbit of the R1-crossing according to $i \in \underline{2}^p$. Let C denote the subcomplex spanned by all the generators of $\coprod_{j \neq 0^p} Kg(\tilde{D}'_j)$ as well as the generators of $Kg(\tilde{D}'_0)$ that do not contain the product $a_1 \dots a_p$, where the a_i are as in Figure 3.10

Some thought shows that the subcomplex C is acyclic. One can see this, by, for example, iterating the usual proof that Reidemeister I moves preserve the chain-homotopy type of $Kc(L)$.

Furthermore, $Kc_o(\tilde{D})$ is naturally identified with $Kc_o(\tilde{D}')/C$. We have a quotient map

$$Kc_o(\tilde{D}') \rightarrow Kc_o(\tilde{D}), \quad (3.84)$$

which is a chain homotopy equivalence (because C is acyclic). This map is induced from a subfunctor inclusion $\mathcal{KHO}(\tilde{D}) \rightarrow \mathcal{KHO}(\tilde{D}')$, in that (3.84) is the dual map on totalizations:

$$\mathrm{Tot}(F_2)^* \rightarrow \mathrm{Tot}(F_1)^*.$$

Here we have used Theorem 3.69 to relate the Khovanov chain complex with the totalizations. That is, we have a (equivariant) stable equivalence $F_1 \rightarrow F_2$; but we have not yet seen that it is an *external* equivariant stable equivalence.

That is, we must also show that the induced map

$$\mathrm{Tot}(F_2^{\mathbb{Z}_q})^* \rightarrow \mathrm{Tot}(F_1^{\mathbb{Z}_q})^*$$

is a homotopy-equivalence for each $q > 1$ dividing p . For this, let $b_1, \dots, b_{p/q}$ denote the images of the Reidemeister circles a_i in the quotient \tilde{D}/\mathbb{Z}_q . Consider the subcomplex E of $AKc_o(\tilde{D}'/\mathbb{Z}_q)$ generated as before by all generators except those of $(\tilde{D}'/\mathbb{Z}_q)_{0^{p/q}}$ that contain the product $b_1 \dots b_{p/q}$. As usual, one checks that E is acyclic, and $AKc_o(\tilde{D}/\mathbb{Z}_q) = AKc_o(\tilde{D}'/\mathbb{Z}_q)/E$, so the map

$$AKc_o(\tilde{D}'/\mathbb{Z}_q) \rightarrow AKc_o(\tilde{D}/\mathbb{Z}_q) \tag{3.85}$$

is a quasi-isomorphism.

Moreover, the subfunctor inclusion $\mathcal{KHO}(\tilde{D}) \rightarrow \mathcal{KHO}(\tilde{D}')$ described above passes to an inclusion on \mathbb{Z}_q -fixed-point functors $\mathcal{KHO}(\tilde{D})^{\mathbb{Z}_q} \rightarrow \mathcal{KHO}(\tilde{D}')^{\mathbb{Z}_q}$. Using the identification in Theorem 3.75, the induced map on totalizations is (3.85). Since we have already seen that (3.85) is a quasi-isomorphism, we have proved invariance under Reidemeister I moves. Keeping track also of the maps induced on even Khovanov homology shows that the inclusion $F_1 \rightarrow F_2$ is an equivariant stable equivalence of Burnside functors with

external action, as needed.

Invariance under equivariant Reidemeister II and III is shown in much the same way. That is, for each acyclic subcomplex or quotient complex “move” in the usual proof of invariance of $\mathcal{KH}\mathcal{O}$, one iterates the move p times to produce an acyclic sub- (resp. quotient) complex which is equivariant, and whose quotient (resp. dual subcomplex) is homotopy-equivalent to the original complex. The sub (quotient) complexes resulting from fixed-point functors can be understood via Theorem 3.75 and the induced maps on the totalization of the fixed-point functors give chain homotopy equivalences as well, since they are the usual maps used in the proof of invariance for the odd annular Khovanov-Burnside functor (without external action).

The proofs of the even version (for all $p > 1$) of the Theorem, as well as the two annular versions, are entirely analogous. \square

Proof of Theorem 3.3. Let $\mathcal{X}_n(\tilde{L})$ denote an equivariant realization modeled on $\tilde{\mathbb{R}}^n$ of the stable Burnside functor with external action $\mathcal{KH}\mathcal{O}(\tilde{L})$ and similarly let $\mathcal{AKH}_n(L)$ be the realization of $\mathcal{AKH}\mathcal{O}(L)$ modeled on $\tilde{\mathbb{R}}^n$. The statement that the actions are well-defined is the combination of Proposition 3.66 with Theorem 3.71 and Theorem 3.69. The fixed-point assertions follow from Theorem 3.75 combined with Lemma 3.59. The gradings can be recovered from Proposition 3.12. \square

3.6.5 Smith inequalities

We now use the results on fixed-point functors from Section 3.6.3 to obtain rank inequalities for Khovanov homology. Let p be prime, and $G = \mathbb{Z}_p$.

Recall that the classical Smith inequality (3.2) for a finite G -CW complex M is

obtained by studying two spectral sequences arising from the *Tate bicomplex*:

$$\begin{aligned} C^{\text{Tate}}(M) &= (C^*(M; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}], d^{\text{Tate}}) \\ &:= (\cdots \xrightarrow{1-\psi} C^*(M; \mathbb{F}_p) \xrightarrow{N(\psi)} C^*(M; \mathbb{F}_p) \xrightarrow{1-\psi} C^*(M; \mathbb{F}_p) \xrightarrow{N(\psi)} \cdots), \end{aligned}$$

where ψ generates the G -action on singular cochains $C^*(M; \mathbb{F}_p)$ and $N(\psi)$ is the norm $1 + \psi + \psi^2 + \cdots + \psi^{p-1}$. The filtration by cohomological degree gives a spectral sequence E^\bullet with $E^1 \cong H^*(M; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}]$ while the filtration by θ -degree gives a spectral sequence converging to $H^*(M^G; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}]$. The assumptions provide sufficient boundedness to conclude that E^\bullet also converges to $H^*(M^G; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}]$, and the rank inequality follows. (For a more detailed exposition, cf. [LT16],[Zha18].)

Theorem 3.86. *For a p -periodic link \tilde{L} for prime p , with quotient link L , and each pair of quantum and (k) -gradings (j, k) , there is a spectral sequence with E^1 -page $AKh^{pj-(p-1)k,k}(\tilde{L}; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}]$ (resp. $AKh_o^{pj-(p-1)k,k}(\tilde{L}; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}]$) converging to $E^\infty \cong AKh^{j,k}(L; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}]$ (resp. $AKh_o^{j,k}(L; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}]$). There is also a spectral sequence with $E^1 \cong Kh(\tilde{L}; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}]$ (resp. $Kh_o(\tilde{L}; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}]$) converging to $E^\infty \cong AKh(L; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}]$ (resp. $AKh_o(L; \mathbb{F}_p) \otimes \mathbb{F}_p[\theta, \theta^{-1}]$).*

Proof. First, consider the case of p odd, and odd annular Khovanov homology. Construct the Tate bicomplex for $\mathcal{X}_n(\tilde{L})$ for odd n (here, viewed as a space, without passing to the suspension spectrum), with the singular differential acting vertically. The column-wise filtration gives a spectral sequence with the desired E^1 -page which converges to the homology of the fixed-point set $\mathcal{X}_n(\tilde{L})^G$, which by Theorem 3.3 is $\mathcal{AKH}_n(L)$. Now, for the case of even Khovanov homology, repeat the above recipe for n even. Since odd Khovanov homology $Kh_o(\tilde{L}; \mathbb{F}_2)$ agrees with $Kh(\tilde{L}; \mathbb{F}_2)$ and $AKh_o(L; \mathbb{F}_2) = AKh(L; \mathbb{F}_2)$, we have also covered the case of p even for odd Khovanov homology (rather trivially).

The proof for the spectral sequences starting in the annular case is entirely analogous.

Finally, for the gradings, note that the spectral sequence splits according to the wedge sum components in the CW-realizations. \square

Corollary 3.87. *Maintain the notation from Theorem 3.86. For each pair of quantum and (k) -gradings (j, k) , the following rank inequalities hold (for vector spaces over \mathbb{F}_p):*

$$\dim AKh^{pj-(p-1)k,k}(\tilde{L}; \mathbb{F}_p) \geq \dim AKh^{j,k}(L; \mathbb{F}_p) \text{ and}$$

$$\dim AKh_o^{pj-(p-1)k,k}(\tilde{L}; \mathbb{F}_p) \geq \dim AKh_o^{j,k}(L; \mathbb{F}_p).$$

We also have the rank inequalities (where each object is the sum over all quantum and (k) -gradings):

$$\dim AKh(\tilde{L}; \mathbb{F}_p) \geq \dim Kh(\tilde{L}; \mathbb{F}_p) \geq \dim AKh(L; \mathbb{F}_p) \geq \dim Kh(L; \mathbb{F}_p) \text{ and}$$

$$\dim AKh_o(\tilde{L}; \mathbb{F}_p) \geq \dim Kh_o(\tilde{L}; \mathbb{F}_p) \geq \dim AKh_o(L; \mathbb{F}_p) \geq \dim Kh_o(L; \mathbb{F}_p).$$

Proof. The AKh -to- Kh inequalities follow from the filtration of the Khovanov complex [Rob13]. The middle inequalities follow from Theorem 3.86. \square

3.6.6 Questions

We conclude with some questions about the construction of equivariant Khovanov spaces. Fix throughout a p -periodic link \tilde{L} with quotient L .

- (q-1) We have not attempted to relate the totalization $\text{Tot}(\mathcal{KH}\mathcal{O})$ (that is, the equivariant odd/even Khovanov complex) with any particular CW chain complex of $\mathcal{X}_n(\tilde{L})$, viewed as a \mathbb{Z}_p -equivariant space (see Remark 3.18). This would be useful to understand in order to relate the constructions of this paper with the equivariant Khovanov homology (or an odd version of same) constructed by Politarczyk

[Pol15]. In more generality, it would be desirable to better understand a \mathbb{Z}_p -equivariant cell decomposition of $\mathcal{X}_n(\tilde{L})$, so that, for example, the space $\mathcal{X}_0(\tilde{L})$ could be related to the space constructed in [BPS18].

- (q-2) A better understanding of the case of even p for the odd Khovanov-Burnside functor $\mathcal{KH}\mathcal{O}$ would be desirable. In particular, the techniques of this paper are sufficient to show that for a given periodic-diagram \tilde{D} of \tilde{L} , the functor $\mathcal{KH}\mathcal{O}(\tilde{D})$ admits a \mathbb{Z}_p -external action. However, it is not immediately clear that this action is a link invariant, and moreover, the resulting external action need not be nonsingular. It is not clear to the authors whether (for $n \geq 1$) Theorem 3.3 (including the statement about fixed points) also holds for even p ; we do not know of a counterexample.
- (q-3) Are there applications of the construction of the present paper to showing that some links are not periodic? Borodzik-Politarczyk-Silvero [BPS18] have obtained such applications; are there further applications that require using the odd theory?
- (q-4) Willis [Wil17] showed that the Khovanov homotopy type of torus links $T(n, m)$ stabilizes as $m \rightarrow \infty$. How does this stabilization interact with the \mathbb{Z}_m -action?

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